Name : ....

Roll No. : ....

Invigilator's Signature : ....

2010-11

MATHEMATICS – I

Time Allotted : 3 Hours *Full Marks* : 70

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

GROUP – A

(Multiple Choice Type Questions)

1. Choose the correct alternatives for any *ten* of the following : $10 \times 1 = 10$

i) If α, β are the roots of the equation $x^2 - 3x + 2 = 0$ then $\begin{bmatrix} 0 & \alpha & \beta \\ \beta & 0 & 0 \\ 1 & -\alpha & \alpha \end{bmatrix}$ is

a) 6 b) $\frac{3}{2}$

c) -6 d) 3.

ii) If $y = e^{ax+b}$, then $(y_5)_0 =$

a) ae^b b) a^5e^b

c) a^be^{ax} d) none of these.

iii) If Rolle's theorem is applied to $f(x) = x\left(x^2 - 1\right)$ in $[0, 1]$, then $C =$

a) 1 b) 0

c) $-\frac{1}{\sqrt{3}}$ d) $\frac{1}{\sqrt{3}}$.

iv) If $u + v = x$, $uv = y$, then $\frac{\partial(u, v)}{\partial(x, y)} =$

a) $u - v$ b) uv

c) $u + v$ d) $\frac{u}{v}$.

v) The value of $\int_{-\pi/2}^{\pi/2} \sin^7 \theta \, d\theta$ is

a) $\frac{6.4.2}{7.5.3.1}$ b) $\frac{\lfloor 6}{\lfloor 7}$

c) 0 d) $\frac{2.(6.4.2)}{7.5.3.1}$.

vi) The sequence $\left\{(-1)^n \frac{1}{n}\right\}$ is

a) convergent b) oscillatory

c) divergent d) none of these.

vii) If $\vec{\alpha} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{\beta} = 2\hat{i} - \hat{k}$, then $(\vec{\alpha} \times \vec{\beta}) \cdot \vec{\alpha}$ is equal to

a) $\hat{i} + \hat{j} + \hat{k}$ b) $\hat{i} + \hat{k}$

c) $\hat{i} - \hat{k}$ d) 0.

viii) The matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is

a) symmetric b) skew-symmetric

c) singular d) orthogonal.

ix) The value of t for which

$\vec{f} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + tz)\hat{k}$ is solenoidal is

a) 2 b) -2

c) 0 d) 1.

x) The distance between two parallel planes $x + 2y - z = 4$ and $2x + 4y - 2z = 3$ is

a) $\frac{5}{\sqrt{24}}$ b) $\frac{5}{24}$

c) $\frac{11}{\sqrt{24}}$ d) none of these.

xi) In the M.V. theorem $f(h) = f(o) + hf'(oh)$; $0 < \theta < 1$ if $f(x) = \frac{1}{1+x}$ and $h = 3$, then value of θ is

a) 1

b) $\frac{1}{3}$

c) $\frac{1}{\sqrt{2}}$

d) none of these.

xii) The series $\sum \frac{1}{np}$ is convergent if

a) $p \geq 1$

b) $p > 1$

c) $p < 1$

d) $p \leq 1$.

GROUP – B

(Short Answer Type Questions)

Answer any *three* of the following. $3 \times 5 = 15$

2. If $y = (x^2 - 1)^n$, then show that

$$(x^2 - 1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0.$$ Hence

find $y_n(0)$.

3. Using M.V.T. prove that

$$x > \tan^{-1} x > \frac{x}{1 + x^2}, \quad 0 < x < \pi/2.$$

4. Show that

$$\begin{bmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{bmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

5. Test the nature of the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots\ldots .$$

6. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors, then show that

$$\left[\,\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\,\right] = [\,a\ b\ c\,]^2.$$

7. If $u=\tan^{-1}\dfrac{x^2-y^2}{x-y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\sin 2u$.

GROUP – C

(Long Answer Type Questions)

Answer any *three* of the following. $3 \times 15 = 45$

8. a) Determine the conditions under which the system of equations $x+y+z=1$, $x+2y-z=b$, $5x+7y+az=b^2$, admits of

i) only one solution

ii) no solution

iii) many solutions.

b) Find the eigenvalues and the corrsponding eigen-vecgors of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

c) Find whether the following series is convergent :

$$\left(\frac{2^2}{1^2} - \frac{2}{1} \right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2} \right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3} \right)^{-3} + \ldots\ldots .$$

9\. a) If $f(x) = x^2$, $g(x) = x^3$ on $[1, 2]$, is Cauchy's mean value theorem applicable ? If so, find ξ.

b) If $I_n = \int \frac{\cos n\theta}{\cos \theta} \, d\theta$, show that

$$(n-1)\left(I_n + I_{n-2} \right) = 2 \sin (n-1)\theta .$$

Hence evaluate $\int \left(4 \cos^2\theta - 3 \right) d\theta$.

c) If $r = \left| \vec{r} \right|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

$$\vec{\nabla}\left(r^n \right) = n r^{n-2} \vec{r} .$$

10\. a) Find $\frac{\partial(u, v)}{\partial(r, \theta)}$, where $u = x^2 - 2y^2$, $v = 2x^2 - y^2$

and $x = r \cos \theta$, $y = \sin \theta$.

b) Verify Green's theorem for $\vec{F} = \left(xy + y^2 \right) \hat{i} + x^2 \hat{j}$ where the curve C in bounded by $y = x$ and $y = x^2$.

c) Evaluate : $\int_0^a \int_0^x \int_0^y x^3 y^2 z \, dz \, dy \, dx$.

11. a) Find the maxima and minima of the function $x^3 + y^3 - 3x + 12y + 20$. Also find the saddle point.

b) State Cayley- Hamilton theorem and verify the same for the matrix $A = \begin{bmatrix} 1 & 2 \\ \cdot 2 & -1 \end{bmatrix}$. Find A^{-1} and A^8.

c) Show that Curl $\vec{\nabla} f = 0$,

where $f(x, y, z) = x^2 y + 2xy + z^2$.

12. a) Given the function $= \left. \begin{array}{ll} \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ = 0, & (x, y) = (0, 0) \end{array} \right\}$

Find from definition $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.
Is $f_{xy} = f_{yx}$?

b) Integrate by Charging the order of integration

$$\int_0^a \int_{x^2/a}^{2a - x} xy \, dy \, dx.$$

c) If $F(p, v, t) = 0$, show that

$$\left(\frac{dp}{dt} \right)_{v \text{ constant}} \times \left(\frac{dv}{dp} \right)_{t \text{ constant}} \times \left(\frac{dt}{dv} \right)_{p \text{ constant}} = -1.$$