MATHEMATICS

SECTION – I

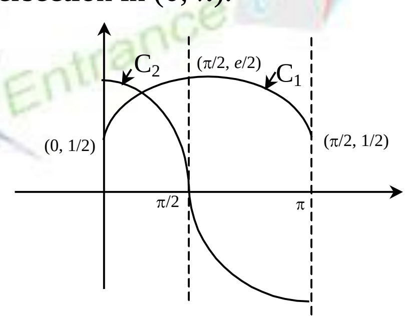
Straight Objective Type

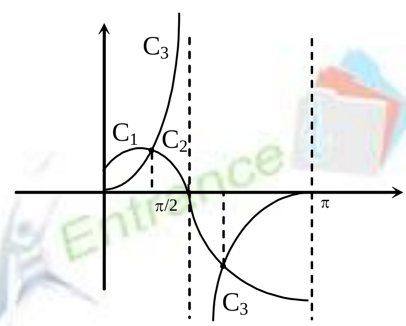
1. **(c)** $A = \left| \int_{\alpha}^{\beta} \left(ax^3 + bx^2 + cx + d - px - q \right) dx \right| = \left| \int_{\alpha}^{\beta} a (x-\alpha)^2 (x-\beta)\, dx \right|$

$= \left| \int_{\alpha}^{\beta} a \left(x^3 - (2\alpha + \beta) x^2 + (\alpha^2 + 2\alpha\beta) x - \alpha^2 \beta \right) dx \right| = \frac{|a|}{12} (\beta - \alpha)^4$

2. **(c)** Let $C_1 \equiv \frac{e^{|\sin x|}}{2}$, $C_2 \equiv \cos x$; $C_3 \equiv \tan x$

We draw graph of y = minimum $\left\{ \frac{e^{|\sin x|}}{2}, \cos x \right\}$ and $y = \tan x$ and find number of points of intersection in $(0, \pi)$.





3. **(c)** $f(x) - f\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^2}\right) + x^2$

$f\left(\frac{x}{2}\right) - f\left(\frac{x}{2^2}\right) = f\left(\frac{x}{2^2}\right) - f\left(\frac{x}{2^3}\right) + \left(\frac{x}{2}\right)^2$

$f\left(\frac{x}{2^2}\right) - f\left(\frac{x}{2^3}\right) = f\left(\frac{x}{2^3}\right) - f\left(\frac{x}{2^4}\right) + \left(\frac{x}{2^2}\right)^2$

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$f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right) = f\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^{n+1}}\right) + \left(\frac{x}{2^{n-1}}\right)^2$

Add and take limit $n \to \infty$

$f(x) - f(0) = f\left(\frac{x}{2}\right) - f(0) + \frac{4x^2}{3} \quad \Rightarrow \quad f(x) - f\left(\frac{x}{2}\right) = \frac{4x^2}{3}$

Again proceed as above, we have

$f(x) - f(0) = \frac{16x^2}{9} \quad (f(0) = 0) \Rightarrow f(x) = \frac{16x^2}{9}$

4. **(b)** $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, also $-1 \le x \le 1$

So, $\tan^{-1} x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

$\therefore \quad k = \dfrac{\pi}{4}, K = \dfrac{3\pi}{4}$

5. **(a)** Required distance is perpendicular distance of Q from tangent at P. Equation of tangent at P is

$$x - ty + at^2 = 0 \qquad \left[\because \ Q: \left(\frac{a}{t^2}, -\frac{2a}{t}\right)\right]$$

Required distance = $\left|\dfrac{\dfrac{a}{t^2} - t \times \left(-\dfrac{2a}{t}\right) + at^2}{\sqrt{1+t^2}}\right| = \dfrac{a}{t^2} \dfrac{(1+t^2)^2}{\sqrt{(1+t^2)}} = \dfrac{a}{t^2}(1+t^2)^{3/2}$

6. **(b)** As $\dfrac{PB}{PA} = 3 \quad \Rightarrow \quad AB = (PB - PA) = 2\sqrt{2} \quad \{\because \ (PA)(PB) = PT^2\}$

As point P is (2, 2) and the given circle $x^2 + y^2 = 2$ has diameter $2\sqrt{2}$.

$\Rightarrow$ AB is diameter of the circle $\Rightarrow$ Line must pass through the centre (0, 0) also.

7. **(b)** Let 'a' and $\dfrac{\tan \frac{B}{2}}{\tan \frac{C}{2}} = k(\text{say})$ given

$$\Rightarrow \quad \left(\frac{s-b}{s-c}\right) = k \quad \Rightarrow \quad \frac{2s-(b+c)}{c-b} = \left(\frac{k+1}{k-1}\right)$$

$(c-b) = \left\{a\left(\dfrac{k-1}{k+1}\right)\right\}$ = constant $\le a(BC)$ $\Rightarrow$ vertex 'A' move on a hyperbola

8. **(c)** $P\left(\dfrac{I_M I_W}{A_M A_W}\right) = \dfrac{4!.(2!)^5}{5!.(2!)^4} = \dfrac{2}{5}$

9. **(b)** We have, $\left|\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right| < \dfrac{\pi}{3} \Rightarrow -\dfrac{\pi}{3} < \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) < \dfrac{\pi}{3}$ $\quad (\because 0 \le \cos^{-1}\theta < \pi)$

$$\Rightarrow \quad 0 \le \cos^{-1}\frac{1-x^2}{1+x^2} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \frac{1-x^2}{1+x^2} \le 1 \Rightarrow 1 + x^2 < 2(1-x^2) \le 2(1+x^2)$$

$$\Rightarrow \quad 0 \le x^2 < \frac{1}{3} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}.$$

Passage – I

10. (b)

11. (a)

12. (c)

Passage – II

13. (c) Equation of plane is $\begin{vmatrix} x-1 & y-1 & z-1 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{vmatrix} = 0$

$\Rightarrow \quad x - 4y + 3z = 0$ …(i)

A plane containing line L_1 is $x - y = 0$ …(ii)

Family of planes containing line of intersection of planes (i) and (ii) is

$x - 4y + 3z + \lambda(x - y) = 0$ …(iii)

14. (d) For no λ, plane is parallel to *x*-*y* plane

15. (a)

16. (a) $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1} \Rightarrow \dfrac{d^2y}{dx^2} = -\left(\dfrac{dx}{dy}\right)^{-2}\left\{\dfrac{d}{dx}\left(\dfrac{dx}{dy}\right)\right\}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-2}\left\{\frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{dx}\right\}$$

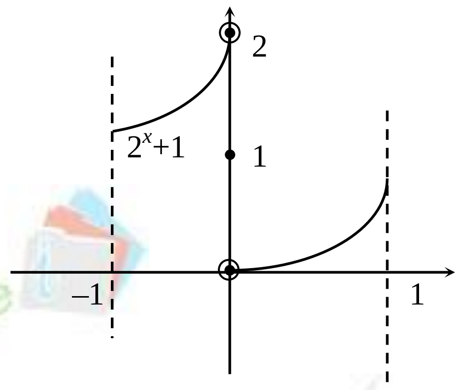
$$= -\left(\frac{dy}{dx}\right)^{2}\left\{\frac{d^2x}{dy^2}.\frac{dy}{dx}\right\} = -\left(\frac{dy}{dx}\right)^{3}\frac{d^2x}{dy^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{3}\frac{d^2x}{dy^2} = 0$$

17. (d) Graph of the function on $[-1, 1]$ is shown below.

$f(x)$ has neither minimum nor maximum value on $[-1, 1]$.

However it is monotonically increasing on $[-1, 1]$.



18. **(b)** $f(x) = 2\sin x + \tan x - 3x$

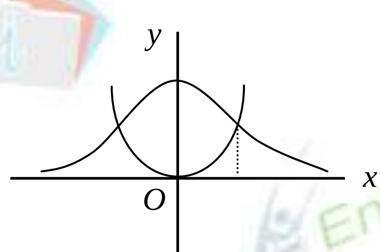
$f'(x) = 2\cos x + \sec^2 x - 3 \Rightarrow \dfrac{\cos x + \cos x + \sec^2 x}{3} \geq 1$

$f(0) = 0;\ f'(x) > 0$ so $f(x) > 0$ for $0 < x < \dfrac{\pi}{2}$ and $f(x) < 0$ for $-\dfrac{\pi}{2} < x < 0$

$\Rightarrow \dfrac{2\sin x + \tan x}{3x} > 1$ for $-\dfrac{\pi}{2} < x < 0,\ 0 < x < \dfrac{\pi}{2}$, thus $\lim\limits_{x \to 0}\left[\dfrac{2\sin x + \tan x}{3x}\right] = 1$

19. **(b)** $x^2 = \dfrac{2}{1+x^2}$

$$A = \int_{-1}^{1}\left(\frac{2}{1+x^2} - x^2\right)dx = 2\left\{\left[2\tan^{-1}x - \frac{x^3}{3}\right]_0^1\right\} = 2\left\{\frac{\pi}{2} - \frac{1}{3}\right\} = \pi - \frac{2}{3}$$



20. **(b)** Considering $P(h, k)$, as $\left(\dfrac{2h-k}{\sqrt{5}}\right)^2 + \left(\dfrac{h-k}{\sqrt{2}}\right)^2 = 1$

$13h^2 + 7k^2 - 18hk - 10 = 0$

Locus is $13x^2 + 7y^2 - 18xy - 10 = 0$ (Here, $\Delta \neq 0$, $H^2 < AB$)

21. (a, b, c, d)	22. (a, b, c)	23. (a, (b, c)	24. (b, c)	25.(a, b, c, d)
26. (a, b, c)	27. (a, b, c, d)	28. (a, b, c, d)	29. (a, b)	30. (a, b)

answers

1 81649
2 201, 499
3 898
4 60/7
5 253
6 25
7 √7250
8 429
9 417/512
10 784

solutions

1. The two digit squares are 16, 25, 36, 49, 64, 81. So if 16 occurs in the number, then it must be followed by 4, which must be followed by 9 and nothing can follow 9. Similarly, 16 can only be preceded by 8, which cannot have predecessor. 64 can also be preceded by 36, which cannot have a predecessor. 25 cannot have successors or predecessors. So the only numbers that cannot be extended at either end are 25, 3649, 81649. The longest is obviously 81649.

2. 601 to 800 study French, 1601 to 1700 study Spanish. So smallest for both is 201 with 400 just French, 1400 just Spanish. Largest is 499, with 301 just French, 1201 just Spanish.

3. We find $a_5 = 267$, $a_6 = -a_1$, $a_7 = -a_2$, $a_8 = -a_3$, $a_9 = -a_4$, $a_{10} = -a_5$, $a_{11} = a_1$, $a_{12} = a_2$, $a_{13} = a_3$, $a_{14} = a_4$ etc. So $a_{531} = a_1$, $a_{753} = a_3$, $a_{975} = a_5$ and $a_1 + a_3 + a_5 = 211 + 420 + 267 = 898$

4. The line through (8,6) parallel to OQ is $10y = 3x + 36$, and this meets OP at (16/7, 30/7). But that must be the midpoint of OP. So P is (32/7, 60/7). The distance to (8,6) is $\sqrt{(24/7)^2 + (18/7)^2} = 30/7$. Hence PQ = 60/7.

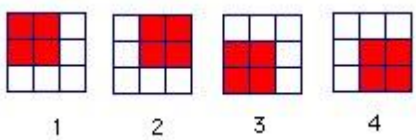
5. Clearly (4, 5, 9, 14, 23, 37, 60, 97, 157, 254) does not have the triangle property (each element from 9 on is the sum of the previous two). But if $a_1 < a_2 < ... < a_{10}$ does not have the property, then $a_{10} \geq a_9 + a_8 \geq 2a_8 + a_7 \geq 3a_7 + 2a_6 \geq ... \geq 34a_2 + 21a_1 \geq 34\cdot5 + 21\cdot4 = 254$.

6. Let the large square have side 2, the small square side x. Then the radius is $\sqrt{2}$. The lines containing vertical sides of the small square pass a distance x from the center. So by Pythagoras, $2 = (1 + 2x)^2 + x^2$, so $(5x-1)(x+1) = 0$, so $x = 1/5$. So ratio $= 1/x^2 = 25$.

7. Let B be the vertex with angle 90^o. Suppose it is a distance x from the points of contact of the incircle. Then chasing around the triangle using the fact that the tangents from a point have equal length, we get $x = (90 + 120 - 150)/2 = 30$. Evidently this is also the radius of the incircle. The other two triangles are similar. The top one has sides 30, 40, 50 and inradius 10, and the right-hand one has sides 45, 60, 75 and inradius 15. We can regard the dotted line as the hypoteneuse of a right-angled triangle with vertical side $60+10-15 = 55$, horizontal side $60+15-10 = 65$. Hence length $\sqrt{55^2 + 65^2} = \sqrt{7250} = 5\sqrt{290}$.

8. The graph of f for $x \in [1,3]$ is piecewise linear with f(1) = f(3) = 0 and a peak at f(2) = 1. The graph for $x \in [3,9]$ is a 3x larger copy with peak at 3, the graph for $x \in [9,27]$ is 3x larger again, with peak at 9 and so on. f(x) is linear from f(2·729) = 729 to f(2001) = 729-543 = 186. f first reaches this value between x = 243 and x = 486. f(243) = 0 and f(486) = 243, so f(x) = 186 at x = 243 + 186 = 429.

9. We use the inclusion/exclusion principle. There are 4 possible red squares as shown. Let $p_{ij...k}$ be the probability of getting i and j and ... and k. Obviously $p_1 = p_2 = p_3 = p_4 = 1/16$, so $\sum p_i = 1/4$. Similarly $p_{12} = p_{13} = p_{24} = p_{34} = 1/64$, $p_{14} = p_{23} = 1/128$, so $\sum p_{ij} = 5/64$. We have $p_{ijk} = 1/256$, so $\sum p_{ijk} = 1/64$. Finally $p_{1234} = 1/512$. Hence p(none) = 1 - 1/4 + 5/64 - 1/64 _ 1/512 = 417/512.



10. $(10^6 - 1) = (10^3 + 1)(10^3 - 1)$, so $10^6 - 1$ and hence also $10^{6n} - 1$ is certainly divisible by 1001. But 7 divides 1001 and $10^m - 1$ is not divisible by 7 for m not a multiple of 6. Obviously 10^m is never divisible by 7. Thus $10^m(10^n - 1)$ is divisible by 1001 iff n is a multiple of 6. So for j = 0, there are 16 values of i (6, 12, ... , 96). Similarly for j = 1, 2, 3. For the next 6 values of j there are 15 values of i, total 6·15. For the next 6, there are 14 each, total 6·14, and so on up to the 6 values 88 to 93 for which there is one each, total 6·1. For j > 93, there are no values of i. Hence total = 64 + 6(15 + 14 + ... + 1) = 64 + 720 = 784.