MATHEMATICS

SECTION – I

Straight Objective Type

1. **(a)** $a \le \sin A \Rightarrow \dfrac{a}{\sin A} \le 1 \Rightarrow 2R \le 1 \Rightarrow R \le \dfrac{1}{2}$

For any point (x, y) inside the circumcircle, $x^2 + y^2 < \dfrac{1}{4}$

$\dfrac{x^2 + y^2}{2} \ge |xy| \Rightarrow |xy| < \dfrac{1}{8}$

2. **(a)** The digit in the units place of $n(n+1)$ is 0, 2 or 6, therefore digit in the units place of number $n(n+1)+1$ is 1, 3, 7.

$\Rightarrow$ 5 doesn't divide $n^2 + n + 1$

$\Rightarrow$ 2005 is not divisor of $n(n+1)+1$

3. **(b)**

4. **(d)**

5. **(d)** $f'(x) = 2(x-1)(2x^2 - x + 2) = 0$

$x = 1$

$f''(x) > 0, \ \forall x \in R$

$x = 1$ is minima

Also $f(0) > 1$, $f(1) = -1$

$\Rightarrow$ two positive and two non-real complex roots.

6. **(a)** Let A be a symmetric matrix, then $A' = A$

$(B'AB)' = B'A'(B')' = B'A'B = B'AB$

7. **(b)**

8. **(c)**

The total number of ways of selecting r letters

From $nA's$, $nB's$ and $nC's$

=coefficient of x^r in $(1 + x + x^2 + ... + x^n)^3$

$=$coefficient of x^r in $\left(\dfrac{1-x^{n+1}}{1-x}\right)^3$

$=$coefficient of x^r in $\left(1-3x^{n+1}+3x^{2n+2}-x^{3n+3}\right)(1-x)^{-3}$

$=$coefficient of x^r in $(1-x)^{-3}$

-3(coefficient of x^{r-n-1} in $(1-x)^{-3}$) + 3(coefficient of x^{r-2n-2} in $(1-x)^{-3}$)

$$=\begin{cases} {}^{r+2}C_2 & ; \quad 1 \le r \le n \\ {}^{r+2}C_2 - 3\,{}^{r-n+1}C_2 & ; \quad n+1 \le r \le 2n+1 \\ {}^{r+2}C_2 - 3\,{}^{r-n+1}C_2 + 3\,{}^{r-2n}C_2 & ; \quad 2n+2 \le r \le 3n \end{cases}$$

9. **(c)**

10. **(a)**

11. **(d)**

12. **(d)** $f(x)-1 = a(x-\alpha_1)(x-\beta_1) = a\left(x^2-(\alpha_1+\beta_1)x+\alpha_1\beta_1\right)$

$f(x) = a\left(x^2-(\alpha_1+\beta_1)x+\alpha_1\beta_1\right)+1$

$f(0) = a\left(0-(\alpha_1+\beta_1)(0)+(-1)\right)+1 = 1-a = \dfrac{3}{2} \Rightarrow a = -\dfrac{1}{2}$

$f(x) = -\dfrac{1}{2}\left(x^2-(\alpha_1+\beta_1)x-1\right)+1$

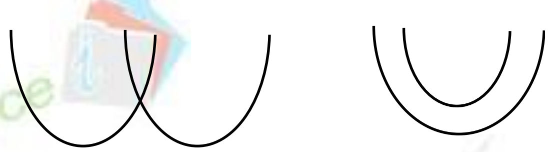
$f''(x) = -1$

13. **(c)** Let $f(x) = a_1x^2+b_1x+c_1$ and $g(x) = a_2x^2+b_2x+c_2$

$\Rightarrow \quad f''(x) = 2a_1$ and $g''(x) = 2a_2$

If $f''(x) = g''(x) \Rightarrow a_1 = a_2$

$f(x) = a_1x^2+b_1x+c_1$ and $g(x) = a_2x^2+b_2x+c_2$ are equal parabola having same concavity.



(intersect at one point) (never intersect)

From the above figure we have that option (a) and (b) are possible but option (c) in never possible.

14. **(a)** $g(x) = b(x-\alpha_2)(x-\beta_2)+1 \quad \ldots$(i)

$f(x) = a(x-\alpha_1)(x-\beta_1)+1 \quad \ldots$(ii)

$g''(x) = 2b$ and $f''(x) = 2a$

$\Rightarrow \; g(x) = \dfrac{g''(x)}{2}(x-\alpha_2)(x-\beta_2)+1 \; \ldots(\text{iii})$

and $f(x) = \dfrac{f''(x)}{2}(x-\alpha_1)(x-\beta_1)+1 \qquad \ldots(\text{iv})$

$g\left(\dfrac{\alpha_2+\beta_2}{2}\right) = -\dfrac{1}{2}\left[g''\left(\dfrac{\alpha_2+\beta_2}{2}\right)\right]\left(\dfrac{\beta_2-\alpha_2}{2}\right)^2 + 1 \quad \ldots(\text{v})$

$f\left(\dfrac{\alpha_1+\beta_1}{2}\right) = -\dfrac{1}{2}\left[f''\left(\dfrac{\alpha_1+\beta_1}{2}\right)\right]\left(\dfrac{\beta_1-\alpha_1}{2}\right)^2 + 1 \quad \ldots(\text{vi})$

Now $f''(x)$ is positive

so $f\left(\dfrac{\alpha_1+\beta_1}{2}\right) = 1 - \dfrac{1}{2}\left[f''\left(\dfrac{\alpha_1+\beta_1}{2}\right)\right]\left(\dfrac{\beta_1-\alpha_1}{2}\right)^2 < 0$

$\Rightarrow \; 1 - \left|g''\left(\dfrac{\alpha_2+\beta_2}{2}\right)\right|\left(\dfrac{\beta_2-\alpha_2}{2}\right)^2 < 0$

$\{\because g''(x)$ is constant and $\beta_2 - \alpha_2 = \beta_1 - \alpha_1)$

Now if $g''\left(\dfrac{\alpha_2+\beta_2}{2}\right)$ is positive then $1 - 2\left(1 - g\left(\dfrac{\alpha_2+\beta_2}{2}\right)\right) < 0$

$\Rightarrow \quad g\left(\dfrac{\alpha_2+\beta_2}{2}\right) < \dfrac{1}{2} \; \ldots(\text{vii})$

If $g''\left(\dfrac{\alpha_2+\beta_2}{2}\right)$ is negative then $1 + 2\left(1 - g\left(\dfrac{\alpha_2+\beta_2}{2}\right)\right) < 0$

$\Rightarrow \; g\left(\dfrac{\alpha_2+\beta_2}{2}\right) > \dfrac{3}{2} \qquad \ldots(\text{viii})$

From (vii) and (viii) we get $g\left(\dfrac{\alpha_2+\beta_2}{2}\right)$ lies between $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{3}{2}, \infty\right)$

15. **(c)** $3^{2x^2} - 2.3^{x^2+x+6} + 3^{2(x+6)} = 0$

$(3^{x^2})^2 - 2(3^{x^2}).3^{x+6} + (3^{x+6})^2 = 0$

$(3^{x^2} - 3^{x+6})^2 = 0 \Rightarrow 3^{x^2} = 3^{x+6} \Rightarrow \; x^2 - x - 6 = 0$

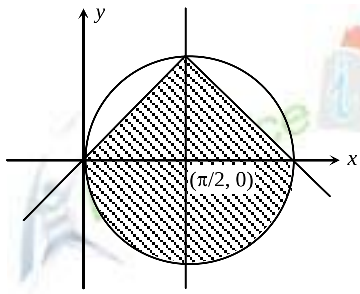
$\Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2, 3$

16. **(c)**

17. **(c)**

18. **(a)** $\left(x - \dfrac{\pi}{2}\right)^2 + y^2 = \left(\dfrac{\pi}{2}\right)^2$

Required area $= \frac{1}{2}\pi\left(\frac{\pi}{2}\right)^2 + \frac{1}{2}.\pi.\frac{\pi}{2} = \frac{\pi^3}{8} + \frac{\pi^2}{4} = \frac{\pi^2}{4}\left(\frac{\pi}{2}+1\right)$ (units)^2



19. (b)

20. (c)

MORE THAN ONE

21. (b, c)	22. (a, d)	23. (a, c)	24. (a, b)	25. (a, c)
26. (b, c)	27. (a, c)	28. (b, c)	29. (a, b)	30. (a, d)

SUBJECTIVE

1 120, 719 9 615

2 101/200 10 83

3 484

4 12

5 $154+40\pi/3$

6 $12+12\sqrt{2}+4\sqrt{3}$

7 380

8 129

solutions

1. ((3!)!)! = 720!, so n = 719.

2. green/$\pi = (2^2-1^2) + (4^2-3^2) + ... + (100^2-99^2) = (2\cdot1+1) + (2\cdot3+1) + ... + (2\cdot99+1) = 50 + 50\cdot100$. So ratio = 101/200.

3. sum = $7\cdot34 + 6\cdot21 + 5\cdot13 + 4\cdot8 + 3\cdot5 + 2\cdot3 + 1\cdot2 = 484$

4. $\sin x \cos x = 1/10$, so $(\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + 1/5 = 6/5$. Hence $\log(\sin x + \cos x) = ½ \log 6/5 = ½ \log 12/10 = (\log 12 - 1)/2$.

5. The box is 60. There are 4 quarter-cylinders length 3 and similarly for the other edges, total $3\pi + 4\pi + 5\pi = 12\pi$. There are 8 one-eighth spheres at the vertices, total $4\pi/3$, and there are boxes height 1 on each face, 12+12+15+15+20+20 = 94. Total $154 + 40\pi/3$

6. There are 24 half-faces, 24 with one edge a long-diagonal, and 8 equilateral (side $\sqrt{2}$). Hence $12 + 12\sqrt{2} + 4\sqrt{3}$.

7. Let M be midpoint of AC. D must lie on perpendicular at M. Suppose DM = x. Note that x does not have to be an integer. Put $k = x^2$. $k+36 = a^2$ and $k+225 = b^2$ are squares. b-a = 1 gives 95^2-94^2. b-a = 3 gives 33^2-30^2. We must have b-a divides $189 = 3^3 7$. b-a = 7 gives 17^2-10^2. b-a = 9 gives 15^2-6^2. b-a = 21 or more does not work. b-a = 9 also does not work because it gives D on AC. So perimeters 2·95+30, 2·33+30, 2·17+30 total 380.

8. a, a+d, a+2d, $(a+2d)^2/(a+d)$. So $(a+2d)^2-a(a+d) = 30(a+d)$ or $3ad + 4d^2 = 30a + 30d$. Hence d is multiple of 3. Also $d(4d-30) = 3a(10-d)$. So $7.5 < d < 10$. Hence d = 9. Hence a = 18. So 18, 27, 36, 48. Sum 129.

9. Suppose the 2-digit sum is d. We must have $1 \le d \le 18$. For $d \le 9$, ther e are d+1 choices for the 2nd pair and one less for the 1st pair (no. cannot have leading 0). For $d > 9$, there are 19-d choices for each pair. Hence $1\cdot2 + 2\cdot3 + ... + 9\cdot10 + 9^2 + 8^2 + ... + 1^2 = 615$.

10. Take X on the line AM so that angle XBC = 7^o. Since CA = CB, X must lie on the angular bisector of $\angle C$. So $\angle BCX = 53^o$. Hence $\angle XCM = 53^o - \angle ACM = 30^o$. Also $\angle XMC = 7^o+23^o = 30^o$. Now XB is perpendicular to MC. But XM = XC, so it must be the perpendicular bisector of MC. Hence $\angle CMB = \angle MCB = 83^o$.