INDIAN INSTITUTE OF SCIENCE
BANGALORE - 560012

ENTRANCE TEST FOR ADMISSIONS - 2009

Program : Integrated Ph.D
Entrance Paper : Mathematical Sciences
Paper Code : MS

Day & Date
SUNDAY, 26TH APRIL 2009

Time
2.00 P.M. TO 5.00 P.M.

Instructions

1. This question paper has forty multiple choice questions.

2. Four possible answers are provided for each question and only one of these is correct.

3. Marking scheme: Each correct answer will be awarded 2.5 marks, but 0.5 marks will be **deducted** for each incorrect answer.

4. Answers are to be marked in the OMR sheet provided.

5. For each question, darken the appropriate bubble to indicate your answer.

6. Use only HB pencils for bubbling answers.

7. Mark only one bubble per question. If you mark more than one bubble, the question will be evaluated as incorrect.

8. If you wish to change your answer, please erase the existing mark completely before marking the other bubble.

9. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers respectively.

10. Let $[x]$ denote the greatest integer less than or equal to x for a real number x.

Integrated Ph. D./ Mathematical Sciences

1. Let T and S be linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let T rotate each point counterclockwise through an angle θ about the origin and let S be the reflection about the line $y = x$. Then determinant of TS is

   (A) 1.

   (B) -1.

   (C) 0.

   (D) 2.

2. Let V be a 7 dimensional vector space. Let W and Z be subspaces of V with dimensions 4 and 5 respectively. Which of the following is not a possible value of $\dim(W \cap Z)$?

   (A) 1.

   (B) 2.

   (C) 3.

   (D) 4.

3. If $a, b \in \mathbb{R}$ satisfy $a^2 + 2ab + 2b^2 = 7$, then the largest possible value of $|a - b|$ is

   (A) $\sqrt{7}$.

   (B) $\sqrt{\frac{7}{2}}$.

   (C) $\sqrt{35}$.

   (D) 7.

4. Suppose a finite group G has an element a which is not the identity such that a^{20} is the identity. Which of the following can not be a possible value for the number of elements of G?

(A) 12.
(B) 9.
(C) 20.
(D) 15.

5. Let A be a 10×10 matrix in which each row has exactly one entry equal to 1, the remaining nine entries of the row being 0. Which of the following is not a possible value for the determinant of the matrix A?

(A) 0.
(B) -1.
(C) 10.
(D) 1.

6. A subset V of $\mathbb{R}^3$ consisting of vectors (x_1, x_2, x_3) satisfying $x_1^2 + x_2^2 + x_3^2 = k$ is a subspace of $\mathbb{R}^3$ if k is

(A) 0.
(B) 1.
(C) -1.
(D) none of the above.

7. Let $v_1 = (1, 0), v_2 = (1, -1)$ and $v_3 = (0, 1)$. How many linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ are there such that $Tv_1 = v_2, Tv_2 = v_3$ and $Tv_3 = v_1$?

(A) 3!.
(B) 3.
(C) 1.
(D) 0.

8. The equation $x^3 + 7x^2 + 1 + ixe^{-x} = 0$ has

(A) no real solution.

(B) exactly one real solution.

(C) exactly two real solutions.

(D) exactly three real solutions.

9. How many complex numbers $z = x + iy$ are there such that $x + y = 1$ and $\exp i(x^2 + y^2) = 1$?

(A) Zero.

(B) Non-zero but finitely many.

(C) Countably infinite.

(D) Uncountably infinite.

10. Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group and suppose it is given that $g_i^2 =$ identity for $i = 1, 2, \ldots, n - 1$. Then

(A) g_n^2 is identity and G is abelian.

(B) g_n^2 is identity, but G could be non-abelian.

(C) g_n^2 may not be identity.

(D) none of the above can be concluded from the given data.

11. Let $X = \{2, 3, 4, \ldots\}$ be the set of integers greater than or equal to 2. Consider the binary relation R on X given by the following: mRn if m and n have a common integer factor $r \neq 1$. Then R is

(A) reflexive and transitive but not symmetric.

(B) reflexive and symmetric but not transitive.

(C) symmetric and transitive but not reflexive.

(D) an equivalence relation.

12. If X and Y are two non-empty finite sets and $f : X \to Y$ and $g : Y \to X$ are mappings such that $g \circ f : X \to X$ is a surjective (i.e., onto) map, then

(A) f must be one-to-one.

(B) f must be onto.

(C) g must be one-to-one.

(D) X and Y must have the same number of elements.

13. Let X and Y be two non-empty sets and let $f: X \to Y$, $g: Y \to X$ be two mappings. If both f and g are injective (i.e., one-to-one) then

(A) X and Y must be infinite sets.

(B) $g = f^{-1}$ always.

(C) one of $f \circ g: Y \to X$ and $g \circ f: X \to Y$ is always bijective (one-to-one and onto).

(D) There exists a bijective mapping $h: X \to Y$.

14. Consider the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

where a_i, b_i, c_i, d_i are real numbers for $1 \leq i \leq 3$. If $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \neq 0$ then the above system has

(A) at most one solution.

(B) always exactly one solution.

(C) more than one but finitely many solutions.

(D) infinitely many solutions.

15. Consider the group $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ of order 16, where the operation is component wise addition modulo 4. If G is a union of n subgroups of order 4 then the minimum value of n is

(A) 4.
(B) 5.
(C) 6.
(D) 7.

16. The altitude of a triangle is a line which passes through a vertex of the triangle and is perpendicular to the opposite side. The orthocenter is the point of intersection of the three altitudes. Let A be the triangle whose vertices are $(1,0)$, $(3,-1)$ and $(0,3)$. Then the orthocenter of A is

(A) $(4/3, 2/3)$.
(B) $(-3,-3)$.
(C) $(-1,1)$.
(D) $(3,5)$.

17. The area of the triangle formed by the straight lines $8x - 3y = 48$, $7y + 4x = 24$ and $5y - 2x = 22$ is

(A) 26.
(B) 30.
(C) 34.
(D) 36.

18. The equation $x^2 - y^2 + (a+b)x + (a-b)y + c = 0$ represents

(A) either a hyperbola or a pair of straight lines.
(B) always a hyperbola.
(C) always a pair of straight lines.
(D) always a parabola.

19. If the volume of the tetrahedron whose vertices are $(1, 1, 1)$, $(3, 2, 0)$, $(0, 4, 3)$ and $(5, 0, k)$ is 6 then the value of k is

(A) $-16/7$.
(B) $-4/7$.
(C) $2/7$.
(D) 2.

20. Which one of the following curves intersects every plane in the 3-dimensional Euclidean space $\mathbb{R}^3$?

(A) $(x, y, z) = (t, t^2, t^3)$.
(B) $(x, y, z) = (t, t^3, t^4)$.
(C) $(x, y, z) = (t, t^3, t^5)$.
(D) $(x, y, z) = (t, t^2, t^5)$.

21. Let $Q = (0, 0, b)$ and $R = (0, 0, -b)$ be two points in the 3-dimensional Euclidean space $\mathbb{R}^3$. If the difference of the distances of a point P in $\mathbb{R}^3$ from Q and R is $2a$ (where $a \neq \pm b$) then the locus of P is

(A) $\frac{x^2}{b^2 - a^2} + \frac{y^2}{b^2 - a^2} - \frac{z^2}{a^2} - 1 = 0$.
(B) $\frac{x^2}{b^2 - a^2} + \frac{y^2}{b^2 - a^2} - \frac{z^2}{a^2} + 1 = 0$.
(C) $\frac{x^2}{a^2 - b^2} + \frac{y^2}{a^2 - b^2} - \frac{z^2}{a^2} + 1 = 0$.
(D) $\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{z^2}{a^2} + 1 = 0$.

22. Define a function f on the real line by

$$f(x) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer,} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

Then which of the following is true:

(A) f is periodic with period 1, i.e., $f(x + 1) = f(x)$ for all x.
(B) f is continuous.
(C) f is one-to-one.
(D) $\lim_{x \to a} f(x)$ exists for all $a \in \mathbb{R}$.

23. Let a, b and c be non-zero real numbers. Let

$$f(x) = \begin{cases} \sin x & \text{if } x \leq c \\ ax + b & \text{if } x > c \end{cases}$$

Suppose b and c are given. Then

(A) There is no value of a for which f is continuous at c.

(B) There is exactly one value of a for which f is continuous at c.

(C) There are infinitely many values of a for which f is continuous at c.

(D) Continuity of f at c can not be determined from what is given.

24. Let

$$f(x) = \begin{cases} 1 \text{ if } |x| \leq 1, \\ 0 \text{ if } |x| > 1 \end{cases} \quad \text{and } g(x) = 2 - x^2.$$

Let $h(x) = f(g(x))$. Then $h(x)$

(A) is continuous everywhere.

(B) has exactly one point of discontinuity.

(C) has exactly two points of discontinuity.

(D) has four points of discontinuity.

25. Let $0 < a < b$. Define a function $M(r)$ for $a \leq r \leq b$ by

$$M(r) = \max\{\frac{r}{a} - 1, 1 - \frac{r}{b}\}.$$

Then $\min\{M(r) : a \leq r \leq b\}$ is

(A) 0.

(B) $2ab/(a + b)$.

(C) $(b - a)/(b + a)$.

(D) $(b + a)/(b - a)$.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) = 1 + xg(x)$ where $\lim_{x\to 0} g(x) = 1$. Then the function $f(x)$ is

(A) e^x,

(B) 2^x,

(C) a non-constant polynomial,

(D) equal to 1 for all $x \in \mathbb{R}$.

27. Let $f(x)$ be a continuous function on $[0, a]$ such that $f(x)f(a-x) = 1$. Then

$$\int_0^a \frac{dx}{1+f(x)}$$

is

(A) 0,

(B) 1,

(C) a,

(D) $a/2$.

28. Let $f : [0, \infty) \to [0, \infty)$ satisfy

$$(f(x))^2 = 1 + 2\int_0^x f(t)dt.$$

Then $f(1)$ is

(A) $\log_e 2$,

(B) 1,

(C) 2,

(D) e.

29. Let

$$f(x) = \int_1^x \frac{e^t}{t} dt$$

for $x \geq 1$. Then $f(x) > \log_e x$

(A) for no value of x.

(B) only for $x > e$.

(C) for $1 \leq x \leq e$.

(D) for all $x > 1$.

30. Consider the first order ODE

$$\frac{dy}{dx} = F\left(\frac{ax+by+c}{Ax+By+C}\right)$$

where a, b, c, A, B and C are non-zero constants. Under what condition, does there exist a linear substitution that reduces the equation to one in which the variables are separable?

(A) Never.

(B) if $aB = bA$.

(C) if $bC = cB$.

(D) if $cA = aC$.

31. Let φ be a solution of the ODE

$$x^2y' + 2xy = 1 \text{ on } 0 < x < \infty.$$

Then the limit of $\varphi(x)$ as $x \to \infty$

(A) is zero.

(B) is one.

(C) is ∞.

(D) does not exist.

32. Let φ be the solution of $y' + iy = x$ such that $\varphi(0) = 2$. Then $\varphi(\pi)$ equals

(A) $i\pi$.

(B) $-i\pi$.

(C) π.

(D) $-\pi$.

33. Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with real entries. Suppose it has repeated eigenvalues. Pick the correct statement:

(A) $bc = 0$.

(B) A is always a diagonal matrix.

(C) $\det(A) \geq 0$.

(D) $\det(A)$ can take any real value.

34. Let G denote the group of all 2×2 real matrices with non-zero determinant. Let H denote the subgroup of all matrices with determinant 1. Let G/H denote the set of left cosets of H. Then

(A) H is not a normal subgroup.

(B) G/H is isomorphic to the real numbers under addition.

(C) G/H is isomorphic to the non-zero real numbers under multiplication.

(D) G/H is a finite group.

35. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors in $\mathbb{R}^3$ such that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$. Then the smaller of the two angles subtended by $\vec{a}$ and $\vec{b}$ is

(A) zero.

(B) an acute angle.

(C) a right angle.

(D) an obtuse angle.

36. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous. Define the function

$$G(x, y) = f(\sqrt{x^2 + y^2}) \text{ for all } (x, y) \in \mathbb{R}^2.$$

Then

(A) $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ are always continuous at each $(x, y) \in \mathbb{R}^2$.

(B) $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ always exist but are not continuous at some point.

(C) G is always continuous on $\mathbb{R}^2$.

(D) The continuity of G depends on the choice of f.

37. The value of the integral

$$\int_0^1 \int_y^1 y\sqrt{1+x^3}\,dx\,dy$$

is

(A) $2\sqrt{2}$.

(B) $(2\sqrt{2}-1)/2$.

(C) $(2\sqrt{2}-1)/8$.

(D) $(2\sqrt{2}-1)/9$.

38. Consider the pair of first order ordinary differential equations

$$\frac{dx}{dt} = Ax + By, \quad \frac{dy}{dt} = x,$$

where $B < -1 < A < 0$. Let $(x(t), y(t))$ be the solution of the above that satisfies $(x(0), y(0)) = (0, 1)$. Pick the correct statement:

(A) $(x(t), y(t)) = (0, 1)$ for all $t \in \mathbb{R}$.

(B) $x(t)$ is bounded on $\mathbb{R}$.

(C) $x(t)$ is bounded on $[0, \infty)$.

(D) $y(t)$ is bounded on $\mathbb{R}$.

39. Let $f(x)$ be a non-constant second degree polynomial such that $f(2) = f(-2)$. If the real numbers a, b and c are in arithmetic progression, then $f'(a), f'(b)$ and $f'(c)$ are

(A) in arithmetic progression.

(B) in geometric progression.

(C) in harmonic progression.

(D) equal.

40. Let $P(x)$ be a non-constant polynomial such that $P(n) = P(-n)$ for all $n \in \mathbb{N}$. Then $P'(0)$

(A) equals 1.

(B) equals 0.

(C) equals -1.

(D) can not be determined from the given data.

End of question paper