MATHEMATICS III & IV

DIPLOMA COURSE IN ENGINEERING

SECOND SEMESTER

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FOREWORD

We take great pleasure in presenting this book of mathematics to the students of Polytechnic Colleges. This book is prepared in accordance with the new syllabus framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind, the aptitude and attitude of the students and modern methods of education. The lucid manner in which the concepts are explained, make the teaching learning process more easy and effective. Each chapter in this book is prepared with strenuous efforts to present the principles of the subject in the most easy-to-understand and the most easy-to-workout manner.

Each chapter is presented with an introduction, definition, theorems, explanation, worked examples and exercises given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that this book serves the purpose i.e., the curriculum which is revised by DTE, keeping in mind the changing needs of the society, to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

List of reference books provided will be of much helpful for further reference and enrichment of the various topics.

We extend our deep sense of gratitude to Thiru.S.Govindarajan, Co-ordinator and Principal, Dr. Dharmambal Government polytechnic College for women, Chennai and Thiru. P.L. Sankar, convener, Rajagopal polytechnic College, Gudiyatham who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and constructive criticisms for improvement of this book will be thankfully acknowledged.

Wishing you all success.

SYLLABUS

SECOND SEMESTER MATHEMATICS - III

UNIT - I

VECTOR ALGEBRA - I

1.1 Introduction: Definition of vectors – types, addition and subtraction of vectors, Properties, of addition and subtraction, position vector, Resolution of vector in two and three dimensions, Direction cosines, direction ratios - Simple Problems.

1.2 Scalar Product of vectors: Definition of scalar product of two vectors – Properties – Angle between two vectors simple problems.

1.3 Application of scalar Product: Geometrical meaning of scalar Product. Work done by Force. Simple Problems

UNIT - II

VECTOR ALGEBRA – II

2.1. Vector product of two vectors:

Definition of vector product of two vectors-Geometrical meaning-properties-angle between two vectors–unit vector perpendicular to two vectors-simple problems.

2.2. Application of vector product of two vectors and Scalar Triple Product:

Definition of moment of a force, definition of scalar product of three vectors - geometrical meaning – coplanar vectors - simple problems.

2.3. Product of more vectors:

Vector Triple Product - Scalar and Vector product of four vectors. Simple problems

UNIT - III

INTEGRATION - I

3.1 Definition of integration – Integral values using reverse process of differentiation – Integration using decomposition method- Simple problems.

3.2 Integration by substitution: Integrals of the form

$\int [f(x)]^n f'(x)dx$ where $n \neq -1$, $\int \frac{f'(x)}{f(x)} dx$,

$\int F[f(x)]\, f'(x)dx$ - Simple Problems

3.3 Standard Integrals

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$, $\int \frac{dx}{\sqrt{a^2 - x^2}}$, $\int \frac{Ax + B}{ax^2 + bx + c} dx$

Simple Problems.

UNIT - IV

INTEGRATION - II

4.1 Integration by Parts

Integrals of the form $\int x \sin nx\, dx$, $\int x \cos nx\, dx$, $\int xe^{nx} dx$, $\int x^n \log x dx$, $\int x \log x\, dx$ - Simple Problems

4.2 Bernoulli's Formula

Evaluation of the integrals $\int x^m \cos nx\, dx$, xm sin nx dx, $\int x^m \int e^{nx} dx$, when $m \geq 2$ using Bernoulli's formula. Simple Problems.

4.3 Definite Integrals

Definition of definite integral, properties of definite integrals. Simple problems.

UNIT - V

PROBABLITY DISTRIBTION – 1

5.1 RANDOM VARIABLE

Definition of Random Variable – Type –Probability Mass Function – Probability density function. Simple problems.

5.2. Mathematical expectation of discrete variable, mean and variance. Simple problems.

5.3 BINOMIAL DISTRIBUTION

Definition

$$P(X = X) = \begin{cases} nCxp^{x}q^{n-x} & , x = 0,,1,,2,......., n \\ 0 & , \text{ Otherwise} \end{cases}$$

(Statement only) Expressions for mean and variance, Simple Problems

SYLLABUS

SECOND SEMESTER MATHEMATICS - IV

UNIT - I

COMPLEX NUMBERS – I

1.1 Definition–Conjugates–Algebra of complex numbers (geometrical proof not needed)–Real and Imaginary parts. Simple problems.

1.2 Polar form of complex number – Modulus and amplitude form multiplication and division of complex numbers in polar form. Simple Problems.

1.3 Argand plane–collinear points, four points forming square, rectangle, rhombus. Simple Problems.

UNIT – II

COMPLEX NUMBER-II

2.1 Demoivre's Theorem (statement only) –simple Problems

2.2 Demoivre's Theorem related Problems. Simple Problems

2.3 Finding the n^{th} roots of unity – Solving equation of the form $x^n \pm 1=0$ where $n \leq 7$ Simple Problems

UNIT- III

PROBABILITY DISTRIBUTION - II

3.1. POISSON DISTRIBUTION

Definition: $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} \quad x = 0,1,2,.......$

(Statement only). Expression for mean and Variance. Simple problems.

3.2. NORMAL DISTRIBUTION

Definition of normal and standard normal distribution. (Statement only). Constants of normal distribution (results only) – Properties of normal distribution – Simple problems using the table standard normal distribution.

3.3. CURVE FITTING

Fitting of a straight line using least square method (result only) – simple problems.

UNIT- IV

APPLICATION OF INTEGRATION AND FIRST ORDER DIFFERENTIAL EQUATION

4.1. AREA AND VOLUME

Area – Area of circle, volume – volume of cone and sphere – simple problems

4.2. FIRST ORDER DIFFERENTIAL EQUATION

Definition of order and degree of differential equation – solution of first order variable separable type differential equation – simple problems

4.3. LINEAR TYPE DIFFERENTIAL EQUATION

Solution of linear differential equation – simple problems

UNIT - V

SECOND ORDER DIFFERENTIAL EQUATIONS

5.1 Solution of second order differential equations with constant coefficients in the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$. Simple Problems

5.2 Solution of second order differential equations in the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ Where a, b and c are constants and $f(x) = e^{mx}$, simple problems.

5.3 Solution of second order differential equations in the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$. Where a,b and c are constants and $f(x) =$ sinmx or cosmx. Simple problems

SECOND SEMESTER

MATHEMATICS - III

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SEMESTER II

MATHEMATICS – III

UNIT – I

VECTOR ALGEBRA - I

1.1 Introduction: Definition of vectors – types, addition and subtraction of vectors, Properties, addition and subtraction, position vector, Resolution of vector in two and three dimensions, Direction cosines, direction ratios - Simple Problems.

1.2 Scalar Product of vectors: Definition of scalar product of two vectors – Properties – Angle between two vectors simple problems.

1.3 Application of scalar Product: Geometrical meaning of scalar Product. Work done by Force. Simple Problems

1.1 INTRODUCTION

A Scalar quantity or briefly a Scalar has magnitude, but is not related to any direction in space. Examples of such are mass volume, density, temperature, work, real numbers.

A vector quantity, or briefly a vector has magnitude and is related to a definite direction in space. Examples of such are displacement, velocity, acceleration, momentum, force etc.

A vector is a directed line segment. The length of the segment is called magnitude of the vector. The direction is indicated by an arrow joining the initial and final points of the line regiment. The vector AB, ie joining the initial point A and the final point B in the direction of AB is denoted as $\overrightarrow{AB}$. The magnitude of the vector $\overrightarrow{AB}$ is AB = $\left|\overrightarrow{AB}\right|$

Zero vector or Null vector:

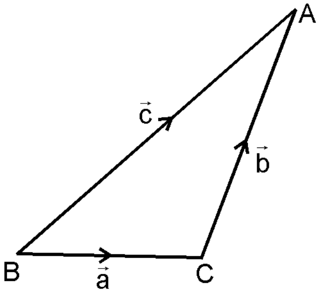
A Zero vector is one whose magnitude is zero, but no definite direction associated with it, for example, if A is a point, $\overrightarrow{AA}$ is a zero vector.

Unit vector: A vector of magnitude one unit is called an unit vector if $\overrightarrow{a}$ is an unit vector, it is also denoted as $\hat{a}$ i.e $|\hat{a}| = |\vec{a}| = 1$.

Negative vector: If $\overrightarrow{AB}$ is a vector, then the negative vector of $\overrightarrow{AB}$ is $\overrightarrow{BA}$. If the direction of a vector changed, we can get the negative vector. i.e $\overrightarrow{BA} = -\overrightarrow{AB}$.

Equal vectors: Two vectors are said to be equal, if they have the same magnitude and the same direction, but it is not required to have the same segment for the two vectors. For example, in a parallelogram ABCD, $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Addition of two vectors: If $\overrightarrow{BC} =, \vec{a}$ $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{BA} = \vec{c}$, then

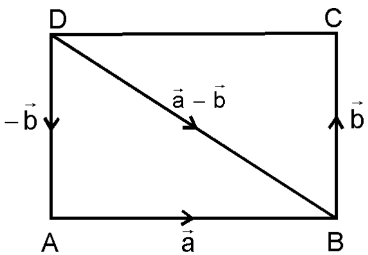


$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$ i.e $\vec{a} + \vec{b} = \vec{c}$ [see figure] If the end point of first vector and the initial point of the second vector are same, the addition of two vectors can be found as the vector joining the initial point of the first vectors and the end point of the second vector.

Properties of vectors addition:

1) vector addition is commutative i.e $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
2) vector addition is associative i.e, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Subtraction of two vectors: if $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$



$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

$= \overrightarrow{AB} + \overrightarrow{CB}$

$= \overrightarrow{AB} + \overrightarrow{DA}$ [$\therefore \overrightarrow{CB}$ an $\overrightarrow{DA}$ are equal]

$= \overrightarrow{DA} + \overrightarrow{AB}$ [$\therefore$addition is commutative]

$= \overrightarrow{DB}$.

Multiplication by a scalar : If $\vec{a}$ is a given vector and λ is a scalar, then $\lambda\vec{a}$ is a vector whose magnitude is $\lambda|\vec{a}|$ and whose direction is the same to that of $\vec{a}$ provided λ is a positive quantity. If λ is negative, $\lambda\vec{a}$ is a vector whose magnitude is $|\lambda||\vec{a}|$ and whose direction is opposite to that of $\vec{a}$.

Properties:

1) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$
2) $m(n\vec{a}) = n(m\vec{a}) = mn\vec{a}$
3) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Collinear vectors: If $\vec{a}$ and $\vec{b}$ are such that they have the same or opposite directions, they are said to be collinear vectors and one is a numerical multiple of the other, i.e $\vec{b} = \kappa\vec{a}$ or $\vec{a} = k\vec{b}$

Resolution of vectors: Let $\vec{a}, \vec{b}, \vec{c}$ be coplanar vectors such that no two vectors are parallel. Then there exists scalars α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$. Similarly, we can get constants (scalars) such that $\vec{a} = \alpha'\vec{b} + \beta'\vec{c}$ and $\vec{b} = \alpha''\vec{c} + \beta''\vec{a}$. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, no three of which are coplanar then there exist scalars λ, β, γ such that. $\vec{d} = \lambda\vec{a} + \beta\vec{b} + \gamma\vec{c}$

Position Vector: If P is any point in the space and O is the origin then $\overrightarrow{OP}$ is called the position vectors of the point P.

Let P be a point in a Plane. Let O be the origin and $\vec{i}$ and $\vec{j}$ be the unit vectors along the x and y axes in that plane. Then if P is (α,β), the position vector of the point P is $\overrightarrow{OP} = \alpha\vec{i} + \beta\vec{j}$

Similarly if P is any point (x,y,z) in the space $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along x,y,z axes in the space then the position vector of the point P is $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$.

The magnitude of $\overrightarrow{OP} = |\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$

Distance between two points: P and Q are two points in the space with co-ordinates $P(x_1,y_1,z_1)$ and Q (x_2, y_2, z_2) then the position vectors are $\overrightarrow{OP} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ .and $\overrightarrow{OQ} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$ .distance

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Direction Cosines and Direction Ratios: Let AB be a straight line making angles α,β,γ with the Co-ordinate axes $X'OX$, $Y'OY$, $Z'OZ$ respectively. Then $\text{Cos}\alpha$, $\text{Cos}\beta$, $\text{Cos}\gamma$ are called the direction cosines of the line AB and denoted by l,m,n. Let OP be parallel to AB and P be (x,y,z) Then OP also makes angles α,β,γ with x,y and z axes. Now, $OP = r = \sqrt{x^2+y^2+z^2}$

$$\text{Then, Cos}\ \alpha = \frac{x}{r}, \text{Cos}\beta = \frac{y}{r}\ \text{and}\ \text{Cos}\gamma = \frac{z}{r}.$$

Now, sum of squares of the direction corines of any straight line is $l^2+m^2+n^2 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 + \left(\frac{z}{r}\right)^2$

$$= \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$$

Note: Let $\hat{n}$ be the unit vectors along OP. Then

$$\hat{n} = \frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|} = \frac{x\vec{i} + y\vec{j} + 2\vec{k}}{r}$$

$$= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

$$= l\vec{i} + m\vec{j} + n\vec{k}$$

Any three numbers p, q, r proportional to the direction cosines of the straight line AB are called the directions ratios of the straight line AB.

1.1 WORKED EXAMPLES
PART – A

1. If Position vectors of the points A and B are $2\vec{i}+\vec{j}-\vec{k}$ and $5\vec{i}+4\vec{j}+3\vec{k}$, find $\left|\overrightarrow{AB}\right|$.

Solution:

Position vector of the point A,

$\overrightarrow{OA}=2\vec{i}+\vec{j}-\vec{k}$ Position vector of the point B, $\overrightarrow{OB}=5\vec{i}+4\vec{j}-3\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$
$$=\left(5\vec{i}+4\vec{j}-3\vec{k}\right)-\left(2\vec{i}+\vec{j}-\vec{k}\right)$$
$$=3\vec{i}+3\vec{j}-2\vec{k}$$
$$\therefore AB=\left|\overrightarrow{AB}\right|=\sqrt{3^2+3^2+(-2)^2}=\sqrt{9+9+4}=\sqrt{22}$$

2. Find the unit vectors along $4\vec{i}-5\vec{j}+7\vec{k}$.

Solution:

Let $\vec{a}=4\vec{i}-5\vec{j}+7\vec{k}$

$$|\vec{a}|=\sqrt{4^2+(-5)^2+7^2}$$
$$=\sqrt{16+25+49}=\sqrt{90}$$
$$\therefore \text{Unit vector along } \vec{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{4\vec{i}-5\vec{j}+7\vec{k}}{\sqrt{90}}$$

3. Find the direction cosines of the vector $2\vec{j}+3\vec{j}-4\vec{k}$

Solution:

Let $\vec{a}=2\vec{j}+3\vec{j}-4\vec{j}$

$$r=|\vec{a}|=\sqrt{2^2+3^2+(-4)^2}$$
$$=\sqrt{4+9++16}$$
$$=\sqrt{29}$$

$\therefore$ Direction cosines of $\vec{a}$ are

$$\text{Cos}\,\alpha=\frac{x}{r}=\frac{2}{\sqrt{29}},\ \text{Cos}\,\beta=\frac{y}{r}=\frac{3}{\sqrt{29}},\ \text{Cos}\,\gamma=\frac{z}{r}=\frac{-4}{\sqrt{29}}$$

4. Find the direction cosines and direction ratios of the vectors $\vec{i}+2\vec{j}-\vec{k}$

Solution:

Let $\vec{a}=\vec{i}+2\vec{j}-\vec{k}$

$$r=\left|\vec{a}\right|=\sqrt{1^2+2^2+(-1)^2}=\sqrt{1+4+1}=\sqrt{6}$$

$\therefore$ Direction cosines are

$$\text{Cos}\,\alpha=\frac{x}{r}=\frac{1}{\sqrt{6}},\quad \text{Cos}\,\beta=\frac{y}{r}=\frac{2}{\sqrt{6}},\quad \text{Cos}\,\gamma=\frac{z}{r}=\frac{-1}{\sqrt{6}}$$

$\therefore$ Direction ratio of $\vec{a}$ is

$$\text{Cos}\alpha:\text{Cos}\beta:\text{Cos}\gamma=\frac{1}{\sqrt{6}}:\frac{2}{\sqrt{6}}:\frac{-1}{\sqrt{6}}$$

$$=1:2:-1$$

5. If the vectors $\vec{a}=2\vec{i}-3\vec{j}$ and $\vec{b}=-6\vec{i}+m\vec{j}$ are collinear, find the value of m

Solution:

Given $\vec{a}=2\vec{i}-3\vec{j}$ and $\vec{b}=-6\vec{i}+m\vec{j}$ are collinear

$$\therefore \vec{a}=t\vec{b}$$

$$2i-3j=t\left(-6\vec{i}+m\vec{j}\right)$$

$$=-6t\vec{i}+mt\vec{j}$$

Comparing coefficients of $\vec{i}$

$$2=-6t\Rightarrow t=-1/3$$

Comparing coefficients of $\vec{j}$,

$$-3=mt \quad \text{i.e,}\ -3=m\left(-1/3\right)\Rightarrow m=9$$

6. If A (2,3,-4) and B (1,0,5) are two points find the direction cosines of the $\overrightarrow{AB}$

Solutions:

Given the points are A (2,3,-4) and B (1,0,5)

Position vectors are $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$\overrightarrow{OB} = \vec{i} + 5\vec{k}$

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(\vec{i} + 5\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 4\vec{k}\right)$$

$$= -\vec{i} - 3\vec{j} + 9\vec{k}$$

$$r = \left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-3)^2 + 9^2}$$

$$= \sqrt{1 + 9 + 81} = \sqrt{91}$$

$\therefore$Direction cosines of $\overrightarrow{AB}$ are

$$\cos\alpha = \frac{-1}{\sqrt{91}},\ \cos\beta = \frac{-3}{\sqrt{91}},\ \cos\gamma = \frac{9}{\sqrt{91}}$$

PART B

1. Show that the points whose position vectors $2\vec{i} + 3\vec{j} - 5\vec{k}, 3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are Collinear.

Solution:

Let $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 5\vec{k}$

$\overrightarrow{OB} = 3\vec{i} + \vec{j} - 2\vec{k}$

$\overrightarrow{OC} = 6\vec{i} - 5\vec{j} + 7\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(3\vec{i} + \vec{j} - 2\vec{k}\right) - \left(2\vec{i} + 3\vec{j} - 5\vec{k}\right)$$

$$= \vec{i} - 2\vec{j} + 3\vec{k}$$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\left(6\vec{i} - 5\vec{j} + 7\vec{k}\right) - \left(3\vec{i} + \vec{j} - 2\vec{j}\right)$

$= 3\vec{i} - 6\vec{j} + 9\vec{k}$

$= 3\left(\vec{i} - 2\vec{j} + 3\vec{k}\right)$

$= 3\overrightarrow{AB}$

i.e, $\overrightarrow{BC} = 3\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel vectors and B is the common point of these two vectors.

$\therefore$ The given points A, B and C are Collinear.

2. Prove that the points A(2,4,-1), B(4,5,1) and C(3,6,-3) form the vertices of a right angled isosceles triangle.

Solution:

Let $\overrightarrow{OA} = 2\vec{i} + 4\vec{j} - \vec{k}$, $\overrightarrow{OB} = 4\vec{i} + 5\vec{j} + \vec{k}$, $\overrightarrow{OC} = 3\vec{i} + 6\vec{j} - 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(4\vec{i} + 5\vec{j} + \vec{k}\right) - \left(2\vec{i} + 4\vec{j} - \vec{k}\right)$

$= 2\vec{i} + \vec{j} + 2\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(3\vec{i} + 6\vec{j} - 3\vec{k}\right) - \left(4\vec{i} + 5\vec{j} + \vec{k}\right)$

$= -\vec{i} + \vec{j} - 4\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(3\vec{i} + 6\vec{j} - 3\vec{k}\right) - \left(2\vec{i} + 4\vec{j} + \vec{k}\right)$

$= \vec{i} + 2\vec{j} - 2\vec{k}$

Now, $AB = \left|\overrightarrow{AB}\right| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9}$

$BC = \left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + 1^2\ (-4)^2} = \sqrt{1+1+16} = \sqrt{18}$

$AC = \left|\overrightarrow{AC}\right| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9}$

$AB = AC = \sqrt{9} = 3$

$AB^2 + AC^2 = 9 + 9 = 18 = BC^2$

$\therefore$ Triangle ABC is an isosceles right angled triangle.

3. Prove that the position vectors $4\vec{i}+5\vec{j}+6\vec{k}, 5\vec{i}+6\vec{j}+4\vec{k}$ and $6\vec{i}+4\vec{j}+5\vec{k}$ form the vertices of an equilateral triangle.

Solution:

Let $\overrightarrow{OA}=4\vec{i}+5\vec{j}+6\vec{k},\ \overrightarrow{OB}=5\vec{i}+6\vec{j}+,4\vec{k}\ \ \overrightarrow{OC}=6\vec{i}+4\vec{j}+5\vec{k}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(5\vec{i}+6\vec{j}+4\vec{k}\right)-\left(4\vec{i}+5\vec{j}+6\vec{k}\right)$$

$$=\vec{i}+\vec{j}-2\vec{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\left(6\vec{i}+4\vec{j}+5\vec{k}\right)-\left(5\vec{i}+6\vec{j}+4\vec{k}\right)$$

$$=\vec{i}-2\vec{j}+\vec{k}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\left(6\vec{i}+4\vec{j}+5\vec{k}\right)-\left(4\vec{i}+5\vec{j}+6\vec{k}\right)$$

$$=2\vec{i}-\vec{j}-\vec{k}$$

$$\text{Now, } AB=\left|\overrightarrow{AB}\right|=\sqrt{1^2+1^2+(-2)^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$BC=\left|\overrightarrow{BC}\right|=\sqrt{1^2(-2)^2+1^2}=\sqrt{1+4+1}=\sqrt{6}$$

$$AC=\left|\overrightarrow{AC}\right|=\sqrt{2^2+(-1)^2+(-1)^2}=\sqrt{4+1+1}=\sqrt{6}$$

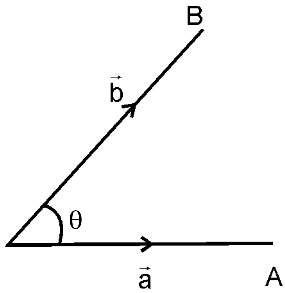
$$\text{Here, } AB=BC=CA=\sqrt{6}$$

$\therefore$ The given points form an equilateral triangle

1.2 SCALAR PRODUCT OF TWO VECTORS OR DOT PRODUCT OF TWO VECTORS

If the product of two vectors $\vec{a}$ and $\vec{b}$ gives a scalar, it is called scalar product of the vectors $\vec{a}$ and $\vec{b}$ and is denoted as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\text{Cos}\,\theta$

Where is the angle between two vectors $\vec{a}$ and $\vec{b}$



Properties of scalar product

1. If θ is an acute angle, $\vec{a}.\vec{b}$ is positive and if θ is an obtuse angle, $\vec{a}.\vec{b}$ is negative.

2. Scalar product is Commutative (i.e) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

3. If $\vec{a}$ and $\vec{b}$ are (non - zero) perpendicular vectors, then $\vec{a}.\vec{b} = 0$

   If $\vec{a}.\vec{b} = 0$, either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a}$ and $\vec{b}$ are perpendicular vector

4. If $\vec{a}$ and $\vec{b}$ are parallel vectors, $\theta = 0°$ or $180°$, $\vec{a}.\vec{b} = ab$

   $\vec{a}.\vec{a} = a^2$

5. $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the x, y and z axes respectively.

   $\therefore\ \vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

   $\vec{i}.\vec{j} = 0 \quad \vec{j}.\vec{i} = 0$

   $\vec{j}.\vec{k} = 0 \quad \vec{k}.\vec{j} = 0$

   $\vec{i}.\vec{k} = 0 \quad \vec{k}.\vec{i} = 0$

Hence,

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors,

$$\vec{a}.\left(\vec{b}+\vec{c}\right)=\vec{a}.\vec{b}+\vec{a}.\vec{c}$$

7. If $\vec{a} = a_1 1\vec{i} + a_2\vec{j} + a_3\vec{k}$ & $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$,

$$\vec{a}.\vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) \bullet \left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$$

$$\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

8. Angle between two vectors

We know, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore \operatorname{Cos}\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \qquad \theta = \cos^{-1}\left[\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

9. $\left(\vec{a}+\vec{b}\right)\bullet\left(\vec{a}+\vec{b}\right)=a^2+b^2+2\vec{a}.\vec{b}.$

10. $\left(\vec{a}-\vec{b}\right)\bullet\left(\vec{a}-\vec{b}\right)=a^2+b^2-2\vec{a}.\vec{b}.$

11. $\left(\vec{a}+\vec{b}\right)\bullet\left(\vec{a}-\vec{b}\right)=a^2-b^2$

1.2 WORKED EXAMPLES

PART A

1. Find the Scalar Product of the two vectors $3\vec{i}+4\vec{j}+5\vec{k}$ and $2\vec{i}+3\vec{j}+\vec{k}$

Solution:

Let : $\vec{a}=3\vec{i}+4\vec{j}+5\vec{k}$

$\vec{b}=2\vec{i}+3\vec{j}+\vec{k}$

$$\bar{a}.\bar{b}=\left(3\vec{i}+4\vec{j}+5\vec{k}\right).\left(2\vec{i}+3\vec{j}+\vec{k}\right)$$

$$=3(2)+4(3)+5(1)$$

$$=6+12+5=23$$

2. Prove that thevectors $3\vec{i}-\vec{j}+5\vec{k}$ and $6\vec{i}+2\vec{j}+4\vec{k}$ are perpendicular.

Solution: Let

$$\vec{a}=3\vec{i}-\vec{j}+5\vec{k},\ \vec{b}=-6\vec{i}+2\vec{j}+4\vec{k}$$

Now $\vec{a}.\vec{b}=\left(3\vec{i}-\vec{j}+5\vec{k}\right).\left(-6\vec{i}+2\vec{j}+\overrightarrow{4k}\right)$

$=3(-6)+(-1)2+5(4)$

$=-18-2+20=0$

$\vec{a}$ and $\vec{b}$ are $\perp^{r}$ vectors

3. Find the value of 'a' if the vectors $2\vec{i}+a\vec{j}-\vec{k}$ and $3\vec{i}+4\vec{j}+2\vec{k}$ are perpendicular.

Solution:

Let $\vec{a}=2\vec{i}+a\vec{j}-\vec{k}$

$\vec{b}=3\vec{i}+4\vec{j}+2\vec{k}$

$\vec{a}$ and $\vec{b}$ are perpendicular.

$\therefore\ \vec{a}.\vec{b}=0$

i.e $\left(2\vec{i}+a\vec{j}-\vec{k}\right).\left(3\vec{i}+4\vec{j}+2\vec{k}\right)=0$

i.e $2(3)+a(4)+(-1)2=0$

i.e $6+4a-2=0$

i.e $4a=2-6=-4$

$$\therefore\ a=-\frac{4}{4}=-1$$

PART B

1. Find the angle between the two vectors $\vec{i}+\vec{j}+\vec{k}$ and $3\vec{i}-\vec{j}+2\vec{k}$

solution:

Let $\vec{a}=\vec{i}+\vec{j}+\vec{k}$, $\vec{b}=3\vec{i}-\vec{j}+2\vec{k}$

$$\vec{a}.\vec{b}=\left(\vec{i}+\vec{j}+\vec{k}\right).\left(3\vec{i}-\vec{j}+2\vec{k}\right)$$

$$=1(3)+1(-1)+1(2)$$

$$=3-1+2=4$$

$$\left|\vec{a}\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

$$\left|\vec{b}\right|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{9+1+4}=\sqrt{14}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\therefore \text{Cos}\,\theta=\frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}=\frac{4}{\sqrt{3}.\sqrt{14}}=\frac{4}{\sqrt{42}}$$

$$\therefore \theta=\text{Cos}^{-1}\left(\frac{4}{\sqrt{42}}\right)$$

2. Show that the vectors $-3\vec{i}+2\vec{j}-\vec{k}, \vec{i}-3\vec{j}+5\vec{j}$ and $2\vec{i}+\vec{j}-4\vec{k}$ form a right angled triangle

Solution:

Let the sides of the triangle be

$\vec{a}=-3\vec{i}+2\vec{j}-\vec{k},\ \vec{b}=\vec{i}-3\vec{j}+5\vec{k},\ \vec{c}=2\vec{i}+2\vec{j}-4\vec{k}$

$$\vec{a}.\vec{b}=\left(-3\vec{i}+2\vec{j}-\vec{k}\right).\left(\vec{i}-3\vec{j}+5\vec{k}\right)$$

$$=-3(1)+(+2)(-3)+(-1)(5)$$

$$=-3-6-5=-14$$

$$\vec{b}.\vec{c}=\left(\vec{i}-3\vec{j}+5\vec{k}\right).\left(2\vec{i}+\vec{j}-4\vec{k}\right)$$

$$=1(2)+(-3)1+5(-4)$$

$$= 2 - 3 - 20 = - 21$$

$$\vec{c}.\vec{a}=\left(2\vec{i}+\vec{j}-4\vec{k}\right).\left(-3\vec{i}+2\vec{j}-\vec{k}\right)$$

$$= 2(-3)+1(+2)+(-4)-(1)$$

$$= - 6 + 2 + 4 = 0$$

$\therefore\ \vec{c}.\vec{a}=0$ implies $\angle A = 90^0$

$\therefore$ The sides $\vec{a},\vec{b}$ and $\vec{c}$ form a right angled triangle

3. Prove that the vectors $2\vec{i}-2\vec{j}+\vec{k},\ \vec{i}+2\vec{j}+2\vec{k},\ 2\vec{i}+\vec{j}-2\vec{k}$ are perpendicular to each other.

Solution:

Let $\vec{a}=2\vec{i}-2\vec{j}+\vec{k}$

$\vec{b}=\vec{i}+2\vec{j}+2\vec{k}$

$\vec{c}=2\vec{i}+\vec{j}-2\vec{k}$

Now, $\vec{a}.\vec{b}=\left(2\vec{i}-2\vec{j}+\vec{k}\right).\left(\vec{i}+2\vec{j}+2\vec{k}\right)$

$$=2(1)+(-2)2+1(2)$$

$$=2-4+2=0$$

$\therefore \vec{a} \perp^r \vec{b}$

$\vec{b}.\vec{c} = \left(\vec{i} + 2\vec{j} + 2\vec{k}\right).\left(2\vec{i} + \vec{j} - 2\vec{k}\right)$

$= 1(2) + 2(1) + 2(-2)$

$= 2 + 2 - 4 = 0.$

$\therefore \bar{b} \perp^r \bar{c}$

$\vec{c}.\vec{a} = \left(2\vec{i} + \vec{j} - 2\vec{k}\right).\left(2\vec{i} - 2\vec{j} + \vec{k}\right)$

$= 2(2) + 1(-2) + (-2)1$

$= 4 - 2 - 2 = 0$

$\therefore \vec{c} \perp^r \vec{a}$

$\therefore$ The three vectors are. $\perp^r$ each another.

1.3 APPLICATION OF SCALAR PRODUCT

Geometrical meaning of scalar product

Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$

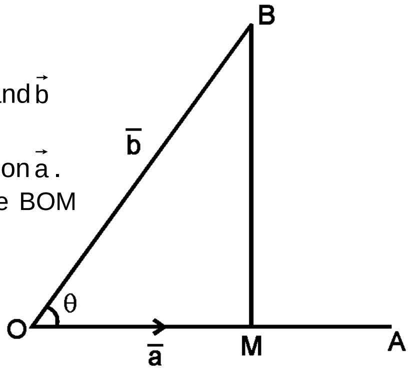
Draw BM perpendicular to OA

Let θ be the angle between $\vec{a}$ and $\vec{b}$

i.e. $B\hat{O}A = \theta$

Now, OM is the projection of $\vec{b}$ on $\vec{a}$.

From the right angled Δ triangle BOM



$$\cos\theta = \frac{OM}{OB} = \frac{OM}{|\vec{b}|}$$

$$\therefore OM = |\vec{b}| \cos\theta$$

$$= \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}|}$$

$$= \frac{\vec{a}.\vec{b}}{|\vec{a}|} \left[\because \text{By definition of Scalar producer}\right]$$

$\therefore$ The Projecton of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

Similary, the projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

WORK DONE:

A force $\vec{F}$ acting on a particle, displaces that particle from the point A to the point B. Hence, the vector $\overrightarrow{AB}$ is called the displacement vector $\vec{d}$ of the particle due to the force $\vec{F}$

The work done $= w = \vec{F}.\vec{d}$.

1.3 WORKED EXAMPLES

PART - A

1. Find the projection of $2\vec{j} + \vec{j} + 2\vec{k}$ on $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution

$$\text{`Let } \vec{a} = 2\vec{j} + \vec{j} + 2\vec{k}, \vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$\text{projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

$$= \frac{\left(2\vec{i} + \vec{j} + 2\vec{k}\right).\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)}{\left|\vec{i} + 2\vec{j} + 2\vec{k}\right|}$$

$$= \frac{2+2+4}{\sqrt{1^2+2^2+2^2}} = \frac{8}{\sqrt{9}} = \frac{8}{3}$$

2. $3\vec{i}+5\vec{j}+7\vec{k}$ is the force acting on a particle giving the displacement $2\vec{i}-\vec{j}+\vec{k}$. Find the workdone.

Solution:

Given $\vec{F}=3\vec{i}+5\vec{j}+7\vec{k}, \vec{d}=2\vec{i}-\vec{j}+\vec{k}$

$$\text{workdone}, w=\vec{F}.\vec{d}$$
$$=\left(3\vec{i}+5\vec{j}+7\vec{k}\right).\left(2\vec{i}-\vec{j}+\vec{k}\right)$$
$$=6-5+7=8$$

PART B

1. A particle moves from the point (1,-2, 5) to the point (3, 4, 6) due to the force $4\vec{i}+\vec{j}-3\vec{k}$ acting on it. Find the work done.

Solution

The force $\vec{F}=4\vec{i}+\vec{j}-3\vec{k}$

The particle moves from A $(1,-2,5)$ to B$(3,4,6)$

$$\therefore \text{Displacement vector } \vec{d}=\overrightarrow{AB}=\vec{OB}-\vec{OA}$$
$$=\left(3\vec{i}+4\vec{j}+6\vec{k}\right)-\left(\vec{i}-2\vec{j}+5\vec{k}\right)$$
$$=2\vec{i}+6\vec{j}+\vec{k}$$

$$\therefore \text{workdone}, W=\vec{F}.\vec{d}$$
$$=\left(4\vec{i}+\vec{j}-3\vec{k}\right).\left(2\vec{i}+6\vec{j}+\vec{k}\right)$$
$$=4(2)+1(6)+(-3)1$$
$$=8+6-3=11 \text{ Units}$$

2. If a particle moves from $3i-\vec{j}+\vec{k}$ to $2\vec{i}-3\vec{j}+\vec{k}$ due to the forces $2\vec{i}+5\vec{j}-3\vec{k}$ and $4\vec{i}+3\vec{j}+2\vec{k}$, find the work done of the forces.

Solution:

The forces are $\vec{F}_1 = 2\vec{i}+5\vec{j}-3\vec{k}$ & $\vec{F}_2 = 4\vec{i}+3\vec{j}+2\vec{k}$.

$\therefore$ Total force $\vec{F} = \overrightarrow{F_1}+\overrightarrow{F_2}$

$$= \left(2\vec{i}+5\vec{j}-3\vec{k}\right)+\left(4\vec{i}+3\vec{j}+2\vec{k}\right) = 6\vec{i}+8\vec{j}-\vec{k}.$$

The particle moves from $\overrightarrow{OA} = 3\vec{i}-\vec{j}+\vec{k}$ to $\overrightarrow{OB} = 2\vec{i}-3\vec{j}+\vec{k}$

$$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$$
$$= \left(2\vec{i}-3\vec{j}+\vec{k}\right)-\left(3\vec{i}-\vec{j}+\vec{k}\right)$$
$$= -\vec{i}-2\vec{j}$$

workdone $= W = \vec{F}.\vec{d}$

$$= \left(6\vec{i}+8\vec{j}-\vec{k}\right).\left(-\vec{i}-2\vec{j}\right)$$
$$= 6(-1)+8(-2)+(-1)0$$
$$= -6-16 = -22 \text{ Units}$$

EXERCISE
PART A

1. If A and B are two points whose position vectors are $\vec{i}-2\vec{j}+2\vec{k}$ and $3\vec{i}+5\vec{j}-7\vec{k}$ respectively find $\overrightarrow{AB}$.
2. If $\overrightarrow{OA} = \vec{i}+2\vec{j}-3\vec{k}$ and $\overrightarrow{OB} = 2\vec{i}-3\vec{j}+\vec{k}$, find $\left|\overrightarrow{AB}\right|$.
3. A and B are (3,2,-1) and (7,5,2) Find $\left|\overrightarrow{AB}\right|$
4. Find the unit vector along $2\vec{i}-\vec{j}+4\vec{k}$
5. Find the unit vector along $\vec{i}+2\vec{j}-3\vec{k}$
6. The position vectors of A and B are $\vec{i}+3\vec{j}-4\vec{k}$ and $2\vec{i}+\vec{j}-5\vec{k}$

   Find the unit vector along $\overrightarrow{AB}$

7. Find the direction cosines of the vector $2\vec{i} - 3\vec{j} + 4\vec{k}$

If $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\overrightarrow{OB} = i + \vec{j} - 2\vec{k}$,

8. Find the direction cosines of the vector $\overrightarrow{AB}$.

9. If A is (2,3,-1) and B is (4,0,7), find the direction ratios of $\overrightarrow{AB}$.

10. Find the modulus and direction cosines of the vector $4\vec{i} - 3\vec{j} + \vec{k}$.

11. Find the direction cosines and direction ratios of the vector $\vec{i} - 2\vec{j} + 3\vec{k}$.

12. If the vectors $\vec{i} + 2\vec{j} + \vec{k}$ and $-2\vec{i} + k\vec{j} - 2\vec{k}$ are collinear, find the value of k.

13. Find the scalar product of the vectors.

(i) $3\vec{i} + 4\vec{j} - 5\vec{k}$ and $2\vec{i} + \vec{j} + \vec{k}$

(ii) $\vec{i} - \vec{j} + \vec{k}$ and $-2\vec{i} + 3\vec{j} - 5\vec{k}$

(iii) $\vec{i} + \vec{j}$ and $\vec{k} + \vec{i}$

(iv) $\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{i} - 2\vec{j} + \vec{k}$

14. Prove that the two vectors are perpendicular to each other.

(i) $3\vec{i} - \vec{j} + 5\vec{k}$ and $-\vec{i} + 2\vec{j} + \vec{k}$

(ii) $8\vec{i} + 7\vec{j} - \vec{k}$ and $3\vec{i} - 3\vec{j} + 3\vec{k}$

(iii) $i - 3\vec{j} + 5\vec{k}$ and $-2\vec{l} + 6\vec{j} + 4\vec{k}$

(iv) $2\vec{i} + 3\vec{j} + \vec{k}$ and $4\vec{l} - 2\vec{j} - 2\vec{k}$

15. If the two vectors are perpendicular find the value of p.

(i) $p\vec{i} + 3\vec{j} + 4\vec{k}$ and $2\vec{i} + 2\vec{j} + -5\vec{k}$

(ii) $p\vec{i} + 2\vec{j} + 3\vec{k}$ and $-\vec{i} + 3\vec{j} - 4\vec{k}$

(iii) $2\vec{i} + p\vec{j} - \vec{k}$ and $3\vec{i} - 4\vec{j} + \vec{k}$

(iv) $\vec{i} + 2\vec{j} - \vec{k}$ and $p\vec{i} + \vec{j}$

(v) $\vec{i} - 2\vec{j} - 4\vec{k}$ and $2\vec{i} - p\vec{j} + 3\vec{k}$

16. Find the projection of

(i) $2\vec{i} + \vec{j} - 2\vec{k}$ on $-\vec{i} - 2\vec{j} - 2\vec{k}$

(ii) $3\vec{i} + 4\vec{j} + 12\vec{k}$ on $\vec{i} + 2\vec{j} + 2\vec{k}$

(iii) $\vec{j}+\vec{k}$ on $\vec{i}+\vec{j}$

(iv) $8\vec{i}+3\vec{j}+\overrightarrow{2k}$ on $\vec{i}+\vec{j}+\vec{k}$

17. Define the scalar product of two vectors $\vec{a}$ and $\vec{b}$
18. Write down the condition for two vectors to be perpendicular.
19. Write down the formula for the projection of $\vec{a}$ and $\vec{b}$
20. If a force $\vec{F}$ acts on a particle giving the displacement $\vec{d}$ write down the formula for the work done by the force.

PART B

1. Prove that the triangle having position vectors of the vectices form an equilateral triangle.

   (i) $4\vec{i}+2\vec{j}+3\vec{k}, 2\vec{i}+3\vec{j}+4\vec{k}, 3\vec{i}+4\vec{j}+2\vec{k}$

   (ii) $3\vec{i}+\vec{j}+2\vec{k}, \vec{i}+2\vec{j}+3\vec{k}, 2\vec{i}+3\vec{j}+\vec{k}$

   (iii) $2\vec{i}+3\vec{j}+5\vec{k}, 5\vec{i}+2\vec{j}+3\vec{k}, 3\vec{i}+5\vec{j}+2\vec{k}$

2. Prove that the following triangle with the vertices form an isosceles triangle.

   (i) $3\vec{i}-\vec{j}-2\vec{k}, 5\vec{i}+\vec{j}-3\vec{k}, 6\vec{i}-\vec{j}-\vec{k}$

   (ii) $-7\vec{j}-10\vec{k}, 4\vec{i}-9\vec{j}-6\vec{k}, \vec{i}-6\vec{j}-6\vec{k}$

   (iii) $7\vec{i}+10\vec{k}, 3\vec{i}-4\vec{j}+6\vec{k}, 9\vec{i}-4\vec{j}+6\vec{k}$

3. Prove that the following position vectors of the vertices of a triangle form a right angled triangle.

   (i) $3\vec{i}+\vec{j}-5\vec{k}, 4\vec{i}+3\vec{j}-7\vec{k}, 5\vec{i}+2\vec{j}-3\vec{k}$

   (ii) $2\vec{i}-\vec{j}+\vec{k}, 3\vec{i}-4\vec{j}-4\vec{k}, \vec{i}-3\vec{j}-5\vec{k}$

   (iii) $2\vec{i}+4\vec{j}+3\vec{k}, 4\vec{i}+\vec{j}-4\vec{k}, 6\vec{i}+5\vec{j}-\vec{k}$

4. Prove that the following vectors are collinear.

   (i) $2\vec{i}+\vec{j}-\vec{k}, 4\vec{i}+3\vec{j}-5\vec{k}, \vec{i}+\vec{k}$

   (ii) $\vec{i}+2\vec{j}+4\vec{k}, 4\vec{i}+8\vec{j}+\vec{k}, 3\vec{i}+6\vec{j}+2\vec{k}$

   (iii) $2\vec{i}-\vec{j}+3\vec{k}, 3\vec{i}-5\vec{j}+\vec{k}, -\vec{i}+11\vec{j}+9\vec{k}$

5. Find the angle between the following two vector

(i) $2\vec{i} - 3\vec{j} + 2\vec{k}$, and, $-\vec{i} + \vec{j} - \vec{k}$

(ii) $4\vec{i} + 3\vec{j} + \vec{k}$, and $2\vec{i} - \vec{j} + 2\vec{k}$,

(iii) $3\vec{i} + \vec{j} - \vec{k}$, and $\vec{i} - \vec{j} - 2\vec{k}$,

6. If the position vectors of A,B and C are $\vec{i} + 2\vec{j} + \vec{k}, 2\vec{i} + 3\vec{k}, 3\vec{i} - \vec{j} + 2\vec{k}$, find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

7. Show that the vectors $\vec{i} - \vec{j} + 2\vec{k}, 4\vec{j} + 2\vec{k}$ and $-10\vec{i} - 2\vec{j} + 4\vec{k}$ are perpendicular to one another.

8. Show that the following position vectors of the points form a right angled triangle

(i) $3\vec{i} - 2\vec{j} + \vec{k}, \vec{i} - 3\vec{j} + 4\vec{k}, 2\vec{i} + \vec{j} - 4\vec{k}$

(ii) $2\vec{i} + 4\vec{j} - \vec{k}, 4\vec{i} + 5\vec{j} - \vec{k}, 3\vec{i} + 6\vec{j} - 3\vec{k}$

(iii) $3\vec{i} - 2\vec{j} + \vec{k}, \vec{i} - 3\vec{j} + 5\vec{k}, 2\vec{i} + \vec{j} - 4\vec{k}$

9. Due to the force $2i - 3\vec{j} + \vec{k}$ a particle is displaced from the point $\vec{i} + 2\vec{j} + 3\vec{k}$ to $-2\vec{i} + 4\vec{j} + \vec{k}$, find the work done.

10. A particle is displaced from A (3, 0, 2) to B (-6,-1,3) due to the force $\vec{F} = 15\vec{i} + 10\vec{j} + 15\vec{k}$, find the work done.

11. $\vec{F} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ displaces a particle from origin to (1,2,-1). Find the work done of the force.

12. Two forces $4\vec{i} + \vec{j} - 3\vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$ displaces a particle from the point (1,2,3) to (5,4,1) find the work done.

13. A Particle is moved from $5\vec{i} + 5\vec{j} - 7\vec{k}$ to $6\vec{i} + 2\vec{j} - 2\vec{k}$ due to the three forces $10\vec{i} - \vec{j} + 11\ \vec{k}$ to $4\vec{i} + 5\vec{j} - 6\vec{k}$ and $-2\vec{i} + \vec{j} - 9\vec{k}$ find the Work done.

14. When a particle is moved from the point (1,1,1) to (2,1,3) by a force $\lambda\vec{i} + \vec{j} + \vec{k}$ the work done is 4. Find the value of λ

15. A force $2\vec{i}+\vec{j}+\lambda\vec{k}$ displaces a particle from the point (1,1,1) to (2,2,2) giving the work done 5. Find the value of.λ

16. Find the value of p, if a force $2\vec{i}-3\vec{j}+4\vec{k}$ displaces a particle from (1,p,3) to (2,0,5) giving the work done 17.

ANSWER

PART A

1. $2\vec{i}+7\vec{j}-9\vec{k}$, 2. $\sqrt{42}$ 3. $\sqrt{34}$,

4. $\dfrac{2\vec{i}-\vec{j}+4\vec{k}}{\sqrt{21}}$ 5. $\dfrac{\vec{i}-2\vec{j}-3\vec{k}}{\sqrt{14}}$, 6. $\dfrac{\vec{i}-2\vec{j}-\vec{k}}{\sqrt{6}}$

7. $\dfrac{2}{\sqrt{29}},\dfrac{-3}{2\sqrt{3}},\dfrac{4}{\sqrt{29}}$ 8. $\dfrac{-1}{3},\dfrac{1}{3},-\dfrac{2}{3}$, 9. $2:-3:8$

10. $\sqrt{26},\dfrac{4}{\sqrt{26}},\dfrac{-3}{\sqrt{26}},\dfrac{1}{\sqrt{26}}$ 11. $\dfrac{1}{\sqrt{14}},\dfrac{-2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}$, $1:-2:3$

12. $K=-4$

13. (i) 5, (ii) -10, (1), (iii)1 (iv) -6

15. (i) 7 (ii) -6, ,(iii)$\dfrac{5}{4}$,(iv) -2 (v) 5

16. (i) $\dfrac{4}{3}$ (ii) $\dfrac{35}{3}$ (iii)$\dfrac{1}{\sqrt{2}}$, (iv)$\dfrac{13}{\sqrt{3}}$

PART B

5. (i) $\mathrm{Cos}^{-1}\left(-\dfrac{7}{\sqrt{51}}\right)$(ii) $\mathrm{Cos}^{-1}\dfrac{7}{234}$ (iii) $\mathrm{Cos}^{-1}\dfrac{4}{\sqrt{66}}$

6. (i) $\mathrm{Cos}^{-1}\left(\dfrac{20}{\sqrt{462}}\right)$ 9. 14, 10. 130

11. 8, 12. 40, 13. 87, 14. -2, 15. 2, 16. $\dfrac{7}{3}$

UNIT- II

VECTOR ALGEBRA – II

2.1. Vector product of two vectors:

Definition of vector product of two vectors-Geometrical meaning-properties-angle between two vectors–unit vector perpendicular to two vectors-simple problems.

2.2. Application of vector product of two vectors and Scalar Triple Product:

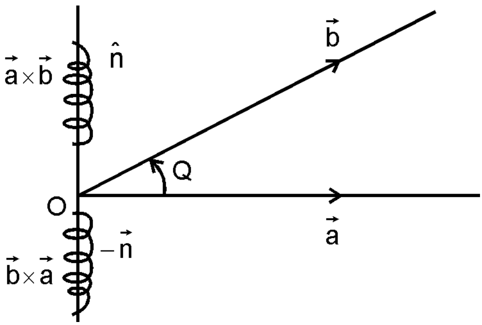
Definition of moment of a force, definition of scalar product of three vectors - geometrical meaning – coplanar vectors - simple problems.

2.3. Product of more vectors:

Vector Triple Product - Scalar and Vector product of four vectors. Simple problems

2.1.VECTOR PRODUCT OF TWO VECTORS OR CROSS PRODUCT

The vector product of two vectors $\vec{a}$ and $\vec{b}$, whose directions are inclined at an angle θ, is the vector whose modulus is $|\vec{a}|\,|\vec{b}|\,\sin\theta$, and whose direction is perpendicular to both $\vec{a}$ and $\vec{b}$, being positive relative to a rotation from $\vec{a}$ to $\vec{b}$.



i.e. $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \vec{n}$

where $\hat{n}$ is an unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, having the same direction as the translation of a right handed screw due to the rotation from $\vec{a}$ to $\vec{b}$. From this it follows that $\vec{b} \times \vec{a}$, has the same length, so that

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

Properties of vector product:

1. $\vec{b} \times \vec{a} = |\vec{b}|\, |\vec{a}| \sin\theta\, (-\hat{n})$

$$= -|\vec{a}|\, |\vec{b}| \sin\theta\, \hat{n}$$

$$= -\vec{a} \times \vec{b}$$

2. If $\vec{a}$ and $\vec{b}$ are parallel, the angle $\theta = 0$

$$\therefore \vec{a} \times \vec{b} = |\vec{a}|\, |\vec{b}| \sin\theta\, \hat{n}$$

$$= |\vec{a}|\, |\vec{b}| \sin 0\, \hat{n}$$

$$= \vec{0} \quad [\because \sin 0 = 0]$$

$\therefore$ The condition for two vectors $\vec{a}$ and $\vec{b}$ to be parallel is $\vec{a} \times \vec{b} = \vec{0}$

In particular, $\vec{a} \times \vec{a} = \vec{0}$

If $\vec{a} \times \vec{b} = 0$, either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel vectors

3. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors,

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

4. If $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along x, y & z axes respectively,

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

Also,
$$\vec{i} \times \vec{j} = -\vec{j} \times \vec{i} = \vec{k}$$
$$\vec{j} \times \vec{k} = -\vec{k} \times \vec{j} = \vec{i}$$
$$\vec{k} \times \vec{i} = -\vec{i} \times \vec{k} = \vec{j}$$

Hence

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

5. If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$,

Then $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Proof:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a} x \vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) \ x \ \left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right)$$

$$= a_1b_1\left(\vec{i} \times \vec{i}\right) + a_1b_2\left(\vec{i} \times \vec{j}\right) + a_1b_3\left(\vec{i} \times \vec{k}\right)$$

$$+ a_2b_1\left(\vec{j} \times \vec{i}\right) + a_2b_2\left(\vec{j} \times \vec{j}\right) + a_2b_3\left(\vec{j} \times \vec{k}\right)$$

$$+ a_3b_1\left(\vec{k} \times \vec{i}\right) + a_3b_2\left(\vec{k} \times \vec{j}\right) + a_3b_3\left(\vec{k} \times \vec{k}\right)$$

$$= \vec{o} + a_1b_2\vec{k} + a_1b_3\left(-\vec{j}\right) + a_2 b_1\left(-\vec{k}\right) + \vec{o}$$

$$+ a_2b_3\vec{i} + a_3b_1\vec{j} + a_3b_2\left(-\vec{i}\right) + a_3 + b_3$$

$$= \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

6. $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$

$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \ldots(1) \qquad [\because |\hat{n}| = 1]$

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}| \sin\theta}$$

$\therefore \hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ by using (1)

7. If 'θ' is the angle between the vectors $\vec{a}$ and $\vec{b}$ then

$$\text{Sin}\,\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

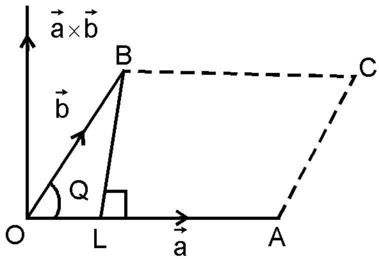
8. **Geometrical meaning of vector product**

Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$

Complete the parallelogram OACB with the sides OA and OB (See figure) Draw BL perpendicular to OA.

Let the angle between triangle $\vec{a}$ and $\vec{b}$ i.e., $A\hat{O}B = \theta$

From right angled triangle OBL,



$$\text{Sin}\,\theta = \frac{BL}{OB}$$

$\therefore BL = OB\,\text{Sin}\theta = |\vec{b}|\text{Sin}\theta$

Now, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\,\text{Sin}\theta$

$= OA.OB\,\text{Sin}\theta$

$= OA.BL$

$= (\text{base}) x (\text{height})$

=Area of the Parallelogram OACB

$\therefore$ If $\vec{a}$ and $\vec{b}$ are the adjacent sides of a parallelogram , the area of the parallelogram= $|\vec{a} \times \vec{b}|$.

9. If $\vec{a}$ and $\vec{b}$ are two sides of a triangle,

area of the triangle $= \frac{1}{2}\left|\vec{a} x \vec{b}\right|$

10. If $\vec{d}_1$ and $\vec{d}_2$ are the diagonal vectors of a parallelogram,

area of the parallelogram $= \frac{1}{2}\left|\vec{d}_1 x \vec{d}_2\right|$

11. If three points A,B and C are collinear

$\overrightarrow{AB} x \overrightarrow{BC} = \overrightarrow{BC} x \overrightarrow{AC} = \overrightarrow{AC} x \overrightarrow{AB} = 0.$

12. If $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ are the position vectors of the vertices of a triangle ABC,

area of the triangle ABC $= \frac{1}{2}\left|\overrightarrow{AB} X \overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{BC} X \overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{AC} X \overrightarrow{AB}\right|$

2.1 WORKED EXAMPLES
PART A

1. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}, \vec{b} = \vec{j} - 2\vec{k}$, find $\vec{a} \text{ x } \vec{b}$

Solution:

$$\vec{a} x \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 0 & 1 & -2 \end{vmatrix} = \vec{i}(-6+1) - \vec{j}(-4+0) + \vec{k}(2-0)$$

$$= -5\vec{i} + 4\vec{j} + 2\vec{k}$$

2. Prove that $\left(\vec{a} \text{ x } \vec{b}\right) \text{x} \left(\vec{a} - \vec{b}\right) = 2\left(\vec{b} - \vec{a}\right)$

Solution:

$$\begin{aligned} L.H.S &= \left(\vec{a} \text{ x } \vec{b}\right) \text{x} \left(\vec{a} - \vec{b}\right) \\ &= \left(\vec{a} \times \vec{a}\right) - \left(\vec{a} \times \vec{b}\right) + \left(\vec{b} \times \vec{a}\right) - \left(\vec{b} \times \vec{b}\right) \\ &= \vec{o} + \left(\vec{b} \times \vec{a}\right) + \left(\vec{b} \times \vec{a}\right) - \vec{o} \\ &= 2\left(\vec{b} \times \vec{a}\right) \\ &= R.H.S \end{aligned}$$

3. Prove that $\vec{a} \times \left(\vec{b}+\vec{c}\right)+\vec{b} \times \left(\vec{c}+\vec{a}\right)+\vec{c} \times \left(\vec{a}+\vec{b}\right)=\vec{o}$

Solution:

$$\begin{aligned}
L.H.S &= \vec{a}\times\left(\vec{b}+\vec{c}\right)+\vec{b}\times\left(\vec{c}+\vec{a}\right)+\vec{c}\times\left(\vec{a}+\vec{b}\right)\\
&= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\\
&= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}\\
&= \vec{0}
\end{aligned}$$

4. Prove that $\vec{i}-2\vec{j}+4\vec{k}$ and $3\vec{i}-6\vec{j}+12\vec{k}$ are parallel vectors.

Solution: Let $\vec{a}=\vec{i}-2\vec{j}+4\vec{k}$

$$\vec{b}=3\vec{i}-6\vec{j}+12\vec{k}$$

Now $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 4\\ 3 & -6 & 12\end{vmatrix}$

$$=3\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & 4\\ 1 & -2 & 4\end{vmatrix} \qquad \frac{R_3}{3}$$

$$=3\left(\vec{0}\right)=\vec{0} \qquad R_2\equiv R_3$$

$\therefore$ The given vectors are parallel.

PART B

1. Prove that $\left(\vec{a}\times\vec{b}\right)^2 + \left(\vec{a}.\vec{b}\right)^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$.

Solution

$$\left(\vec{a}x\vec{b}\right)^2 + \left(\vec{a}.\vec{b}\right)^2$$

$$=\left[\left|\vec{a}\right|\left|\vec{b}\right| \text{ Sin}\theta\,\hat{n}\right]^2+\left[\left|\vec{a}\right|\left|\vec{b}\right| \text{ Cos}\theta\right]^2$$

$$=\left|\vec{a}\right|^2\left|\vec{b}\right|^2 \text{Sin}^2\theta\left(\hat{n}\right)^2+\left|\vec{a}\right|^2\left|\vec{b}\right|^2\text{Cos}^2\theta$$

$$=\left|\vec{a}\right|^2\left|\vec{b}\right|^2\left[\text{Sin}^2\theta+\text{Cos}^2\theta\right]$$

$$=\left|\vec{a}\right|^2\left|\vec{b}\right|^2 \quad .1$$

$$=\left|\vec{a}\right|^2\left|\vec{b}\right|^2$$

$$= \text{R.H.S}$$

2 Find the unit vector perpendicular to $2\vec{i}-\vec{j}+\vec{k}$ and $3\vec{i}+4\vec{j}-\vec{k}$ find also the sine of the angle between these vectors.

Solution

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \vec{i}(1-4)-\vec{j}(-2-3)+\vec{k}(8+3)$$

$$= -3\vec{i}+5\vec{j}+11\vec{k}$$

$$\left|\vec{a}\times\vec{b}\right| = \sqrt{(-3)^2+5^2+11^2}$$

$$= \sqrt{9+25+121} = \sqrt{155}$$

$$\left|\vec{a}\right| = \sqrt{2^2(-1)^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$\left|\vec{b}\right| = \sqrt{3^2+4^2+(-1)^2} = \sqrt{9+16+1} = \sqrt{26}$$

The unit vector $\perp^r$ to both $\vec{a}$ and $\vec{b} = \dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} = \dfrac{-3\vec{i}+5\vec{j}+11\vec{k}}{\sqrt{155}}$

$$\sin\theta = \frac{\left|\vec{a}\times\vec{b}\right|}{\left|\vec{a}\right|\left|\vec{b}\right|} = \frac{\sqrt{155}}{\sqrt{6}.\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$$

3. Find the area of the parallelogram whose adjacent sides are $\vec{i}+\vec{j}+\vec{k}$ and $3\vec{i}-\vec{k}$

Solution

Let the adjacent sides of the parallelogram be

$\vec{a} = \vec{i}+\vec{j}+\vec{k}$ and $\vec{b} = 3\vec{i}-\vec{k}$

$$\vec{a}x\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}(-1-0) - \vec{j}(-1-3) + \vec{k}(0-3)$$

$$= -\vec{i}+4\vec{j}-3\vec{k}$$

Area of the Parallelogram= $\left|\vec{a}\times\vec{b}\right|$

$$= \sqrt{(-1)^2+4^2+(-3)^2}$$

$$= \sqrt{1+16+9} = \sqrt{26} \quad \text{sq.Units}$$

4. Find the area of the triangle whose vertices are having the position vectors $2\vec{i}+3\vec{j}+4\vec{k}$, $3\vec{i}+4\vec{j}+2\vec{k}$ and $4\vec{i}+2\vec{j}+3\vec{k}$.

Solution:

The position vectors of the vertices of the triangle be $\overrightarrow{OA}=2\vec{i}+3\vec{j}+4\vec{k}$

$$\overrightarrow{OB}=3\vec{i}+4\vec{j}+2\vec{k}$$

$$\overrightarrow{OC}=4\vec{i}+2\vec{j}+3\vec{k}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(3\vec{i}+4\vec{j}+2\vec{k}\right)-\left(2\vec{i}+3\vec{j}+4\vec{k}\right)$$

$$=\vec{i}+\vec{j}-2\vec{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\left(4\vec{i}+2\vec{j}+3\vec{k}\right)-\left(3\vec{i}+4\vec{j}+2\vec{k}\right)$$

$$=\vec{i}-2\vec{j}+\vec{k}$$

$$\overrightarrow{AB}x\overrightarrow{BC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 1 & -2\\ 1 & -2 & 1\end{vmatrix}=\vec{i}(1-4)-\vec{J}(1+2)+\vec{k}(-2-1)$$

$$=-3\vec{i}-3\vec{J}-3\vec{k}$$

$$\text{Area of the triangle ABC}=\frac{1}{2}\left|\overrightarrow{AB}X\overrightarrow{BC}\right|$$

$$=\frac{1}{2}\sqrt{(-3)^2+(-3)^2+(-3)^2}$$

$$=\frac{1}{2}\sqrt{9+9+9}=\frac{\sqrt{27}}{2}\quad \text{sq.Units}$$

5.Prove that the position vectors of the points $\vec{i}-2\vec{j}+3\vec{k}$, $2\vec{i}+3\vec{j}-4\vec{k}$ and $-7\vec{j}+10\vec{k}$ form collinear points.

Solution:

Let the position vectors of the three points A,B,C be

$$\overrightarrow{OA}=\vec{i}-2\vec{j}+3\vec{k}$$

$$\overrightarrow{OB}=2\vec{i}+3\vec{j}-4\vec{k}$$

$$\overrightarrow{OC}=-7\vec{j}+10\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{i} + 3\vec{j} - 4\vec{k}) - (\vec{i} - 2\vec{j} + 3\vec{k})$$
$$= \vec{i} + 5\vec{j} - 7\vec{K}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-7\vec{j} + 10\vec{k}) - (2\vec{i} + 3\vec{j} - 4\vec{k})$$
$$= 2\vec{i} - 10\vec{j} + 14\vec{K}$$
$$\overrightarrow{AB} x \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -7 \\ -2 & -10 & 14 \end{vmatrix}$$
$$= (-2)\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -7 \\ 1 & 5 & -7 \end{vmatrix}$$
$$= (-2)\vec{O}$$
$$= \vec{O} \qquad [\because R_2 \equiv R_3]$$

$\therefore$ The points A,B,C are collinear points

6. If $\vec{i} - \vec{j} - 3\vec{k}$ and $2\vec{i} - \vec{j} - 3\vec{k}$ are the diagonals of a parallelogram, find the area of the parallelogram.

Solution:

The diagonals of the parallelogram are $\vec{d}_1 = \vec{i} - \vec{j} - 3\vec{k}, \vec{d}_2 = 2\vec{i} - \vec{j} - 3\vec{k}$

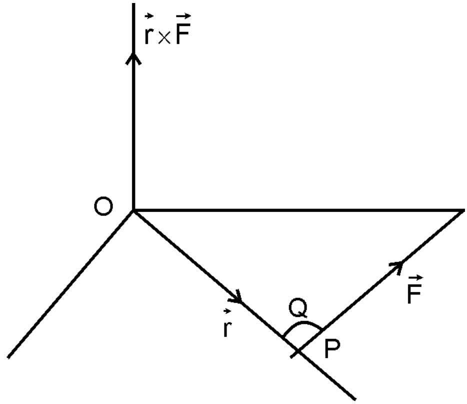
$$\vec{d}_1 x \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -3 \\ 2 & -1 & -3 \end{vmatrix}$$
$$= \vec{i}(3-3) - \vec{j}(-3+6) + \vec{k}(-1+2)$$
$$= \vec{i}(0) - 3\vec{j} + \vec{k} = -3\vec{j} + \vec{k}$$

$\therefore$ Area of the parallelogram

$$= \frac{1}{2}\left|\vec{d}_1 x \vec{d}_2\right|$$
$$= \frac{1}{2}\sqrt{(-3)^2 + 1^2} = \frac{1}{2}\sqrt{9+1} = \frac{\sqrt{10}}{2} \text{ sq.units}$$

2.2 APPLICATION OF VECTOR PRODUCT OF TWO VECTORS

Moment or Torque of a force about a point



Let O be any point and $\vec{r}$ be the position vector relative to the point O of any point P on the line of action of the force $\vec{F}$. The moment of the force about the point O is defined as

$$\vec{M} = \vec{r}\times\vec{F}$$

The magnitude of the moment

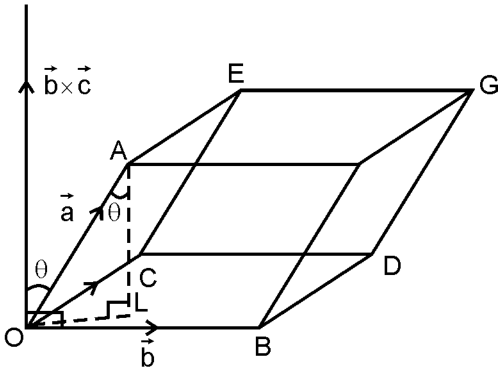
$$|\vec{M}| = |\vec{r}\times\vec{F}|$$

$$= |\vec{r}||\vec{F}| \ \text{Sin}\theta$$

Scalar Triple product of the three vectors:

Scalar triple product of three vectors is defined as $\vec{a}.(\vec{b}\times\vec{c})$ It is denoted as $\vec{a}\bullet(\vec{b}\times\vec{c}) = [\vec{a},\vec{b},\vec{c}]$. It is called box product of three vectors $\vec{a},\vec{b},\text{and}\,\vec{c}$.

Let $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}, \vec{c} = \overrightarrow{OC}$ be the three vectors not lying in the same plane and meeting at the point O.



Complete the parallelopiped with $\vec{a},\vec{b},\vec{c}$ as the adjacent edges with a common intersection. Let AL be the height of the parallelepiped. $\vec{b}\times\vec{c}$ is the perpendicular vector of the parallelogram OBDC $\therefore \left|\vec{b}\times\vec{c}\right|$ is area of the parallelogram OBDC.

Let θ be the angle between $\vec{a}$ and $\left(\vec{b}\times\vec{c}\right)$.

$$\text{Cos}\theta = \frac{AL}{\left|\overrightarrow{OA}\right|} = \frac{AL}{\left|\vec{a}\right|}$$

$$\therefore AL = \left|\vec{a}\right|\cos\theta$$

From the right triangle AOL $\quad \therefore \vec{a}.\left(\vec{b}\times\vec{c}\right) = \left|\vec{a}\right|\left|\vec{b}\times\vec{c}\right|\text{Cos}\theta$

$$= \left|\vec{a}\right|\text{Cos}\theta\left|\vec{b}\times\vec{c}\right|$$

$$= AL\left|\vec{b}\times\vec{c}\right|$$

=(Height of the parallelepiped)$\times$(Area of the base parallelopiped)

$\therefore\ \vec{a}.\left(\vec{b}\times\vec{c}\right)$=Volume of the parallelopiped.

Properties of scalar Triple Product

1. If $\vec{a},\vec{b},\vec{c}$ are the three vectors of a parallelopiped (three edges meeting at a common point),

Volume of the parallelopiped, $V = \vec{a}.\left(\vec{b}\times\vec{c}\right)$

$$= \vec{b}.\left(\vec{c}\times\vec{a}\right) \quad \ldots(1)$$

$$= \vec{c}.\left(\vec{a}\times\vec{b}\right)$$

2. Scalar product is commutative,

$$\vec{a}\bullet\left(\vec{b}\times\vec{c}\right) = \left(\vec{b}\times\vec{c}\right)\bullet\vec{a}$$

$$\vec{b}\bullet\left(\vec{c}\times\vec{a}\right) = \left(\vec{c}\times\vec{a}\right)\bullet\vec{b} \quad \ldots(2)$$

$$\vec{c}\bullet\left(\vec{a}\times\vec{b}\right) = \left(\vec{a}\times\vec{b}\right)\bullet\vec{c}$$

$\therefore$ From (1) and (2) $\vec{a}\bullet\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right)\bullet\vec{c}$

$\therefore$ In scalar triple product , $\bullet$ and x may be interchanged.

3. In a box product of three vectors, if any two of the vectors are equal or parallel to each other, then the scalar triple product is zero.

i.e, $\left[\vec{a},\vec{a},\vec{b}\right] = \left[\vec{a},\vec{c},\vec{c}\right] = \left[\vec{b},\vec{c},\vec{b}\right] = 0$

If $\vec{a}$ is parallel to $\vec{b}$, then $\left[\vec{a},\vec{b},\vec{c}\right] = 0$

4. In the box product of three vectors for each interchange of two vectors, the sign will change.

5. $\left[\vec{i},\vec{j},\vec{k}\right] = \vec{i}.\left(\vec{j}x\vec{k}\right) = \vec{j}.\left(\vec{k}x\vec{i}\right) = \vec{k}.\left(\vec{i}x\vec{j}\right)$

$$= \vec{i}\,\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k}$$

$$= 1.$$

6. If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$$\left[\vec{a},\vec{b},\vec{c}\right] = \vec{a}\bullet\left(\vec{b}x\vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{Proof : } \vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \vec{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \vec{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\therefore\ \vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right).\left[\vec{i}\begin{vmatrix} b_2 & b_3 \\ c_1 & c_3 \end{vmatrix} - \vec{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \vec{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$

$$= a\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

7. If $\vec{a}, \vec{b}, \vec{c}$ are Coplanar vectors (those vectors in the same plane), $\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$

8. If $\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$ then
   i. any one of the three vectors is zero or
   ii. any two of the three vectors are parallel or
   iii. $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

2.2 WORKED EXAMPLES

PART A

1. Find the value of $\left[\vec{i}, \vec{j}, \vec{k}\right]$

Solution:

$$\left[\vec{i}, \vec{j}, \vec{k}\right] = \vec{i}.\left(\vec{j} \times \vec{k}\right)$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 0 + 0$$

$$= 1\ (-0) = 1$$

2. Find the value of $\left[\vec{i}+\vec{j}, \vec{j}+\vec{k}, \vec{k}+\vec{i}\right]$

Solution:

$$\left[\vec{i}+\vec{j}, \vec{j}+\vec{k}, \vec{k}+\vec{i}\right] = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-1) + 0 \qquad = 1+1 = 2$$

3. Find the scalar triple product of the vectors $\vec{i} - 3\vec{j} + 3\vec{k}, 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{j} + \vec{k}$

Solution:

Let $\vec{a} = \vec{i} - 3\vec{j} + 3\vec{k}$

$\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{c} = \vec{j} + \vec{k}$

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 1 & -3 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\ (1+1) + 3(2-0) + 3(2-0)$$

$$= 2 + 6 + 6 \ = 14$$

4. Find the volume of the parallelepiped whose three edges meeting at a point are $2\vec{i} - 3\vec{j} + 4\vec{k}, \vec{i} + 2\vec{j} - \vec{k}, 3\vec{i} - \vec{j} + 2\vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

Volume of the parallelepiped $V = \left[\vec{a}, \vec{b}, \vec{c}\right] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$$= 2(4-1) + 3(2+3) + 4(-1-6)$$

$$= 2(3) + 3(5) + 4(-7)$$

$$= 6 + 15 - 28 = -7$$

$\therefore$ Volume of the parallelepiped =7 cubic Units

PART-B

1. Find the moment of the force $3\vec{i}+\vec{k}$ acting along the point $\vec{i}+2\vec{j}-\vec{k}$ about the point $2\vec{i}+\vec{j}+2\vec{k}$.

Solution:

Let $\vec{F}=3\vec{i}+\vec{k}$

$$\overrightarrow{OA}=\vec{i}+2\vec{j}-\vec{k}$$

$$\overrightarrow{OB}=2\vec{i}-\vec{j}+2\vec{k}$$

$$\vec{r}=\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}$$

$$=\left(\vec{i}+2\vec{j}-\vec{k}\right)-\left(2\vec{i}-\vec{j}+2\vec{k}\right)$$

$$=-\vec{i}+3\vec{j}-3\vec{k}$$

Moment $\vec{M}=\vec{r}\times\vec{F}$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & -3 \\ 3 & 0 & 1 \end{vmatrix}$$

$$=\vec{i}(3+0)-\vec{j}(-1+9)+\vec{k}(0-9)$$

$$=3\vec{i}+\left(-8\vec{j}\right)-9\vec{k}$$

$$=3\vec{i}-8\vec{j}-9\vec{k}$$

$$\therefore \overrightarrow{M}=3\vec{i}-8\vec{j}-9\vec{k}$$

$$\left|\overrightarrow{M}\right|=\sqrt{3^2+(-8)^2+(-9)^2}$$

$$=\sqrt{9+64+81}=\sqrt{154} \text{ units}$$

2. O,A,B,C are points(0,0,0),(1,-2,3),(2,3,4) and (-1,0,2). Find the volume of the Parallelepiped whose edges are OA, OB, and OC.

Solution:

O is the origin

$$\therefore \overrightarrow{OA}=\vec{i}-2\vec{j}+3\vec{k}$$

$$\overrightarrow{OB}=2\vec{i}+3\vec{j}+4\vec{k}$$

$$\overrightarrow{OC}=-\vec{i}+2\vec{k}$$

OA,OB,OC are the edges.

$\therefore$ Volume of the parallelepiped $V = [OA, OB, OC]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & 4 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= 1(6-0) + 2(4+4) + 3(0+3)$$

$$= 6 + 16 + 9 = 31 \text{ Cubic Units.}$$

3. Prove that the vectors $3\vec{i} + 2\vec{j} - 2\vec{k}, 5\vec{i} - 3\vec{j} + 3\vec{k},$ and $5\vec{i} - \vec{j} + \vec{k}$ are coplanar vectors.

Solution:

Let $\vec{a} = 3\vec{i} + 2\vec{j} - 2\vec{k}$

$\vec{b} = 5\vec{i} - 3\vec{j} + 3\vec{k}$

$\vec{c} = 5\vec{i} - \vec{j} + \vec{k}$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 3 & 2 & -2 \\ 5 & -3 & 3 \\ 5 & -1 & 1 \end{vmatrix}$$

$$= 3(-3+3) - 2(5-15) - 2(-5+15)$$

$$= 0 - 2(-10) - 2(10)$$

$$= 20 - 20 = 0$$

$[\vec{a}, \vec{b}, \vec{c}] = 0 \Rightarrow \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

4. If the three vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k},$ and $3\vec{i} + m\vec{j} + 5\vec{k},$ are coplanar, find the value of m.

Solution:

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{c} = 3\vec{i} + m\vec{j} + 5\vec{k}$

since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors then $\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$

$$\text{i.e,} \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$$

$$\text{i.e,} \quad 2(10+3m)+1(5+9)+1(m-6)=0$$

$$\text{i.e,} \quad 20+6m+14+m-6=0$$

$$\text{i.e,} \quad 7m+28=0$$

$$\text{i.e,} 7M=-28$$

$$\text{i.e,} m=-\frac{28}{7}=-4.$$

5. Show that the points whose position vectors are $4\vec{i}+5\vec{j}+\vec{k}, -\vec{j}-\vec{k}, 3\vec{i}+9\vec{j}+4\vec{k}$ and $-4\vec{i}+4\vec{j}+4\vec{k}$ lie on the same plane.

Solution:

Let the position vectors of the four points be

$$\overrightarrow{OA} = 4\vec{i}+5\vec{j}+\vec{k}$$

$$\overrightarrow{OB} = -\vec{j}-\vec{k}$$

$$\overrightarrow{OC} = 3\vec{i}+9\vec{j}+4\vec{k}$$

$$\overrightarrow{OD} = -4\vec{i}+4\vec{j}+4\vec{K}$$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$$

$$= \left(-\vec{j}-\vec{k}\right)-\left(4\vec{i}+5\vec{j}+\vec{k}\right)$$

$$= -4\vec{i}-6\vec{j}-2\vec{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA}$$

$$= \left(3\vec{i}+9\vec{j}+4\vec{k}\right)-\left(4\vec{i}+5\vec{j}+\vec{k}\right)$$

$$= -\vec{i}+4\vec{j}+3\vec{k}$$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \left(-4\vec{i} + 4\vec{j} + 4\vec{k}\right) - \left(4\vec{i} + 5\vec{j} + \vec{k}\right)$

$= -8\vec{i} - \vec{j} + 3\vec{k}$

Now, $\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$

$= -4(12+3) + 6(-3+24) - 2(1+32)$

$= -4(15) + 6(21) - 2(33)$

$= -60 + 126 - 66$

$= 0$

$\therefore \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar vectors.

$\therefore$ The given four points A,B,C and D lie on a same plane.

6. Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$

Solutions:

$$\text{LHS} = \left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right]$$

$$= \left(\vec{a}+\vec{b}\right) \bullet \left[\left(\vec{b}+\vec{c}\right) \times \left(\vec{c}+\vec{a}\right)\right]$$

$$= \left(\vec{a}+\vec{b}\right) \bullet \left[\left(\vec{b}\times\vec{c}\right) + \left(\vec{b}\times\vec{a}\right) + \left(\vec{c}x\vec{c}\right) + \left(\vec{c}\times\vec{a}\right)\right]$$

$$= \left(\vec{a}+\vec{b}\right) \bullet \left[\left(\vec{b}\times\vec{c}\right) + \left(\vec{b}\times\vec{a}\right) + \vec{0} + \left(\vec{c}\times\vec{a}\right)\right]$$

$$= \vec{a} \bullet \left(\vec{b}\times\vec{c}\right) + \vec{a} \bullet \left(\vec{b}x\vec{a}\right) + \vec{a} \bullet \left(\vec{c}x\vec{a}\right) + \vec{b} \bullet \left(\vec{b}x\vec{c}\right) + \vec{b} \bullet \left(\vec{b}x\vec{a}\right) + \vec{b} \bullet \left(\vec{c}x\vec{a}\right)$$

$$= \left[\vec{a},\vec{b},\vec{c}\right] + \left[\vec{a},\vec{b},\vec{a}\right] + \left[\vec{a},\vec{c},\vec{a}\right] + \left[\vec{b},\vec{b},\vec{c}\right] + \left[\vec{b},\vec{b},\vec{a}\right] + \left[\vec{b},\vec{c},\vec{a}\right]$$

$$= \left[\vec{a},\vec{b},\vec{c}\right] + 0 + 0 + 0 + 0\left[\vec{b},\vec{c},\vec{a}\right]$$

$$= \left[\vec{a},\vec{b},\vec{c}\right] + \left[\vec{a},\vec{b},\vec{c}\right] \qquad \left[\because \left[\vec{b},\vec{c},\vec{a}\right] = \left[\vec{a},\vec{b},\vec{c}\right]\right]$$

$$= 2\left[\vec{a},\vec{b},\vec{c}\right].$$

2.3 PRODUCT OF MORE VECTORS

Vector Triple Product: Vectors Triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ or $\left(\vec{a} \times \vec{b}\right) \times \vec{c}$

Results

1. $\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \left(\vec{a}.\vec{c}\right)\vec{b} - \left(\vec{b}.\vec{c}\right)\vec{a}$
2. $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$
3. $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

Product of four vectors

Scalar Product of four vectors:

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, the scalar product of four vectors is $\left(\vec{a} \times \vec{b}\right) \bullet \left(\vec{c} \times \vec{d}\right)$

Here, $\vec{a} x \vec{b}$ is a vector, $\vec{c} x \vec{d}$ is a vector and $\left(\vec{a} x \vec{b}\right) \bullet \left(\vec{c} x \vec{d}\right)$ is the dot product of two vectors and hence it is a scalar quantity.

Result: $\left(\vec{a} x \vec{b}\right) \bullet \left(\vec{c} x \vec{d}\right) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

Proof: $\left(\vec{a} x \vec{b}\right) \bullet \left(\vec{c} x \vec{d}\right) = \left(\vec{a} x \vec{b}\right) \vec{x}$ $\qquad \left[\vec{x} = \vec{c} x \vec{d}\right]$

$$= \vec{a} \bullet \left(\vec{b} x \vec{x}\right)$$

$$= \vec{a} \bullet \left[\vec{b} x \left(\vec{c} x \vec{d}\right)\right]$$

$$= \vec{a} \bullet \left[\left(\vec{b}.\vec{d}\right)\vec{c} - \left(\vec{b}.\vec{c}\right)\vec{d}\right]$$

$$= \left(\vec{a}.\vec{c}\right)\left(\vec{b}.\vec{d}\right) - \left(\vec{a}.\vec{d}\right)\left(\vec{b}.\vec{c}\right)$$

$$= \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}.$$

Vector product of four vectors:

If $\vec{a}, \vec{b}, \vec{c},$ and $\vec{d}$ are four vector, $\vec{a} x \vec{b}$ is a vector, $\vec{c} x \vec{d}$ is a vector and $\left(\vec{a} x \vec{b}\right) x \left(\vec{c} x \vec{d}\right)$ is a vector and it is called vector product of four vectors.

Results:

1. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}, \vec{b}, \vec{d}\right]\vec{c} - \left[\vec{a}, \vec{b}, \vec{c}\right]\vec{d}$

Proof:

Let $\vec{a} x \vec{b} = \vec{x}$

$$\begin{aligned}
\text{L.H.S} &= \left(\vec{a} x \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) \\
&= \vec{x}\ x \left(\vec{c} \times \vec{d}\right) \\
&= \left(\vec{x} \bullet \vec{d}\right)\vec{c} - \left(\vec{x} \bullet \vec{c}\right)\vec{d} \\
&= \left\{ \left(\vec{a} \times \vec{b}\right) \bullet \vec{d} \right\} \vec{c} - \left\{ \left(\vec{a} \times \vec{b}\right) \bullet \vec{c} \right\} \vec{d} \\
&= \left[\vec{a}, \vec{b}, \vec{d}\right] \vec{c} - \left[\vec{a}, \vec{b}, \vec{c}\right] \vec{d} \\
&\text{R.H.S}
\end{aligned}$$

2. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are Coplanar vectors, then $\vec{a} x \vec{b}$ and $\vec{c} x \vec{d}$ are parallel vector perpendicular to the plane of these four vectors.

$$\therefore \left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \vec{o}$$

2.3 WORKED EXAMPLES
PART – A

1. Find the value of $\vec{i} \times \left(\vec{j} \times \vec{k}\right)$

Solution:

$$\begin{aligned}
\vec{i} x \left(\vec{j} \times \vec{k}\right) &= \vec{i} \times \vec{i} \\
&= \vec{o}
\end{aligned}$$

2. .If $\vec{a} = \vec{i} + \vec{j}, \vec{b} = \vec{j} + \vec{k}, \vec{c} = \vec{k} + \vec{i}$ find $\vec{a} x (\vec{b} x \vec{c})$.

$$\vec{b} x \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}(1-0) - \vec{j}(0-1) + \vec{k}(0-1)$$

$$= \vec{i} + \vec{j} - \vec{k}$$

$$\vec{a} x (\vec{b} x \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(-1-0) - \vec{j}(-1-0) + \vec{k}(1-1)$$

$$= -\vec{i} + \vec{j}$$

PART B

1. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}, \quad \vec{b} = \vec{i} - 2\vec{j}, \vec{c} = 2\vec{i} - \vec{j} + \vec{k},$ prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Solution:

$$\vec{a} = \vec{i} - \vec{j} + \vec{k}$$

$$\vec{b} = \vec{i} - 2\vec{j}$$

$$\vec{c} = 2\vec{i} - \vec{j} + \vec{k}$$

$$\vec{b} x \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \vec{i}(-2-0) - \vec{j}(1-0) + \vec{k}(-1+4)$$

$$= -2\vec{i} - \vec{j} + 3\vec{k}$$

$$\vec{a}x(\vec{b}x\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -2 & -1 & 3 \end{vmatrix}$$

$$= \vec{i}(-3+1) - \vec{j}(3+2) + \vec{k}(-1-2)$$

$$(i.e)\ \vec{a}\ x\ (\vec{b}x\vec{c}) = -2\vec{i} - 5\vec{j} - 3\vec{k} \qquad ...(1)$$

$$\vec{a}.\vec{c} = (\vec{i} - \vec{j} + \vec{k}).(2\vec{i} - \vec{j} + \vec{k})$$

$$= 1(2) + (-1)(-1) + 1(1)$$

$$= 2 + 1 + 1 = 4$$

$$\vec{a}.\vec{b} = (\vec{i} - \vec{j} + \vec{k}).(\vec{i} - 2\vec{j})$$

$$= 1(1) + (-1)(-2) + 1(0)$$

$$= 1 + 2 = 3$$

$$\therefore (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$= 4(\vec{i} - 2\vec{j}) - 3(2\vec{i} - \vec{j} + \vec{k})$$

$$= 4\vec{i} - 8\vec{j} - 6\vec{i} + 3\vec{j} - 3\vec{k}$$

$$= -2\vec{i} - 5\vec{j} - 3\vec{k} \qquad ...(2)$$

From (1) & (2) , $\vec{a}x(\vec{b}x\vec{c}) = (\vec{a}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{c}$

2. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - \vec{k}, \vec{c} = \vec{i} + 2\vec{j}, \vec{d} = \vec{i} - \vec{j} - 3\vec{k}$

Find $(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$

Solution:

$$\vec{a} = \vec{i} - \vec{j} + \vec{k}$$

$$\vec{b} = -\vec{i} + 2\vec{j} - \vec{k}$$

$$\vec{c} = \vec{i} + 2\vec{j}$$

$$\vec{d} = \vec{i} - \vec{j} - 3\vec{k}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -1 & 2 & -1 \end{vmatrix}$$

$$= \vec{i}(1-2) - \vec{j}(-1+1) + \vec{k}(2-1)$$

$$= -\vec{i} + \vec{k}$$

$$\vec{c}\times\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= \vec{i}(-6-0) - \vec{j}(-3-0) + \vec{k}(-1-2)$$

$$= -6\vec{i} + 3\vec{j} - 3\vec{k}$$

$$\left(\vec{a}\times\vec{b}\right)\bullet\left(\vec{c}\times\vec{d}\right) = \left(-\vec{i} + \vec{k}\right)\bullet\left(-6\vec{i} + 3\vec{j} - 3\vec{k}\right)$$

$$= (-1)(-6) + 0(3) + 1(-3) \qquad \ldots(1)$$

$$= 6 - 3 = 3$$

Alternate Method

$$\vec{a}.\vec{c} = \left(\vec{i} - \vec{j} + \vec{k}\right)\bullet\left(\vec{i} + 2\vec{j}\right)$$

$$= 1(1) + (-1)2 + 1(0)$$

$$= 1 - 2 \;\; = -1$$

$$\vec{a}.\vec{d} = \left(\vec{i} - \vec{j} + \vec{k}\right)\bullet\left(\vec{i} - \vec{j} - 3\vec{k}\right)$$

$$= 1(1) + (-1)(-1) + 1(-3)$$

$$= 1 + 1 - 3 \;\; = -1$$

$$\vec{b}.\vec{c} = \left(-\vec{i} + 2\vec{j} - \vec{k}\right)\bullet\left(\vec{i} + 2\vec{j}\right)$$

$$= (-1)1 + 2(2) + (-1)0$$

$$= -1 + 4 \;\; = 3$$

$$\vec{b}.\vec{d} = \left(-\vec{i} + 2\vec{j} - \vec{k}\right)\bullet\left(\vec{i} - \vec{j} - 3\vec{k}\right)$$

$$= (-1)1 + 2(-1) + (-3)$$

$$= -1 - 2 + 3 = 0$$

$$\therefore \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 3 & 0 \end{vmatrix} = 0 + 3 = 3$$

3. If $\vec{a} = \vec{i} + \vec{j} + \vec{k},\ \vec{b} = \vec{i} - \vec{j} - \vec{k},\ \vec{c} = -\vec{i} + \vec{j} + 2\vec{k},$

$\vec{d} = 2\vec{i} + \vec{j}$, find $\left(\vec{a}x\vec{b}\right)x\left(\vec{c}x\vec{d}\right)$.

Solution:

$$\vec{a} = \vec{i} + \vec{j} + \vec{k},$$

$$\vec{b} = \vec{i} - \vec{j} - \vec{k},$$

$$\vec{c} = -\vec{i} + \vec{j} + 2\vec{k},$$

$$\vec{d} = 2\vec{i} + \vec{j}$$

$$\vec{a}x\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(-1-1) + \vec{k}(-1-1)$$

$$= \vec{i}(0) - \vec{j}(-2) + \vec{k}(-2)$$

$$= 2\vec{j} - 2\vec{k}$$

$$\vec{c}x\vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= \vec{i}(0-2) - \vec{j}(0-4) + \vec{k}(-1-2)$$

$$= -2\vec{i} + 4\vec{j} - 3\vec{k}$$

$$\therefore \left(\vec{a}x\vec{b}\right)x\left(\vec{c}x\vec{d}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ -2 & 4 & -3 \end{vmatrix}$$

$$= \vec{i}(-6+8) - \vec{j}(0-4) + \vec{k}(0+4)$$

$$= 2\vec{i} + 4\vec{j} + 4\vec{k}$$

Alternate Method

$$\left[\vec{a},\vec{b},\vec{d}\right]=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$=1(0+1)-1(0+2)+1(1+2)$$

$$=1(1)-2+(3)$$

$$=1-2+3=2$$

$$\left[\vec{a},\vec{b},\vec{c}\right]=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$=1(-2+1)-1(2-1)+1(1-1)$$

$$=-1-1=-2$$

$$\therefore \left[\vec{a},\vec{b},\vec{d}\right]\vec{c}-\left[\vec{a},\vec{b},\vec{c}\right]\vec{d}$$

$$=2\left(-\vec{i}+\vec{j}+2\vec{k}\right)-(-2)\left(2\vec{i}+\vec{j}\right)$$

$$=-2\vec{i}+2\vec{j}+4\vec{k}+4\vec{i}+2\vec{j}=2\vec{i}+4\vec{j}+4\vec{k}$$

3. Prove that $\left[\vec{a}x\vec{b},\vec{b}x\vec{c},\vec{c}x\vec{a}\right]=\left[\vec{a},\vec{b},\vec{c}\right]^2$

Solution:

$$\left(\vec{b}x\vec{c}\right)x\left(\vec{c}x\vec{a}\right)=\left[\vec{b},\vec{c},\vec{a}\right]\vec{c}-\left[\vec{b},\vec{c},\vec{c}\right]\vec{a}$$

$$=\left[\vec{b},\vec{c},\vec{a}\right]\vec{c}\quad -0$$

$$=\left[\vec{b},\vec{c},\vec{a}\right]\vec{c}\qquad \left[\because \left[\vec{b},\vec{c},\vec{c}\right]=0\right.$$

$$\left[\vec{a}x\vec{b},\vec{b}x\vec{c},\vec{c}x\vec{a}\right]=\left(\vec{a}x\vec{b}\right)\bullet\left\{\left(\vec{b}x\vec{c}\right)x\left(\vec{c}x\vec{a}\right)\right\}$$

$$=\left(\vec{a}x\vec{b}\right)\bullet\left[\vec{b},\vec{c},\vec{a}\right]\vec{c}$$

$$=\left[\vec{b},\vec{c},\vec{a}\right]\left(\vec{a}x\vec{b}\right).\vec{c}$$

$$=\left[\vec{b},\vec{c},\vec{a}\right]\left[\vec{a},\vec{b},\vec{c}\right]$$

$$=\left[\vec{a},\vec{b},\vec{c}\right]\left[\vec{a},\vec{b},\vec{c}\right]\qquad \left[\because \left[\vec{a},\vec{b},\vec{c}\right]=\left[\vec{b},\vec{c},\vec{a}\right]\right]$$

$$=\left[\vec{a},\vec{b},\vec{c}\right]^2$$

EXERCISE

PART - A

1. Find the vector product of two vectors.

   (i) $2\vec{i}+3\vec{j}-4\vec{k}$ and $\vec{i}-2\vec{j}+4\vec{k}$

   (ii) $\vec{i}+\vec{j}+\vec{k}$ and $2\vec{i}-\vec{j}+3\vec{k}$

   (iii) $2\vec{i}-3\vec{j}+5\vec{k}$ and $\vec{i}-2\vec{j}-2\vec{k}$

   (iv) $3\vec{i}+\vec{j}+\vec{k}$ and $2\vec{i}-3\vec{j}+2\vec{k}$

2. Prove that the two vectors $5\vec{i}-7\vec{j}+3\vec{k}$ and $15\vec{i}-21\vec{j}+9\vec{k}$ are parallel vectors.

3. If $\vec{a}=2\vec{i}+\vec{j}-\vec{k}$ and $\vec{b}=\vec{i}-2\vec{j}+2\vec{k}$ Prove that $\left|\vec{a}x\vec{b}.\right|=5\sqrt{2}$

4. Find the area of the Parallelogram whose adjacent sides are

   (i) $2\vec{i}-3\vec{j}$ and $\vec{i}-2\vec{j}-3\vec{k}$

   (ii) $3\vec{i}+2\vec{j}+2\vec{k}$ and $\vec{i}-2\vec{j}+3\vec{k}$

   (iii) $2\vec{i}+\vec{j}-2\vec{k}$ and $\vec{i}+2\vec{j}+3\vec{k}$

5. Find the Scalar triple product of

   (i) $\vec{i}-2\vec{j}+3\vec{k}, 2\vec{i}+\vec{j}-\vec{k}, \vec{j}+\vec{k}$

   (ii) $2\vec{i}+5\vec{j}+\vec{k}, \vec{i}-2\vec{k}, 5\vec{i}+2\vec{j}-\vec{k}$

   (iii) $\vec{i}-\vec{j}+\vec{k}, 2\vec{i}+3\vec{j}-3\vec{k}, 6\vec{i}-2\vec{j}-\vec{k}$

6. Find the Volume of parallelepiped whose edges are

   (i) $2\vec{i}-4\vec{j}+5\vec{k}, \vec{i}-\vec{j}+\vec{k}, 3\vec{i}-5\vec{j}+2\vec{k}$

   (ii) $2\vec{i}+3\vec{j}+4\vec{k}, 4\vec{i}+3\vec{j}+\vec{k}, \vec{i}+2\vec{j}+4\vec{k}$

   (iii) $3\vec{i}+7\vec{j}+2\vec{k}, 2\vec{i}+5\vec{j}-\vec{k}, \vec{i}+6\vec{j}+\vec{k}$

7. Prove the following three vectors are coplanar

(i) $-\vec{i}+4\vec{j}-3\vec{k}$, $3\vec{i}+2\vec{j}+5\vec{k}$, $-3\vec{i}+8\vec{j}-5\vec{k}$

(ii) $3\vec{i}+2\vec{j}-2\vec{k}$, $5\vec{i}-3\vec{j}+3\vec{k}$, $5\vec{i}-\vec{j}+\vec{k}$

(iii) $\vec{i}-2\vec{j}+3\vec{k}$, $-2\vec{i}+3\vec{j}-4\vec{k}$, $-\vec{j}+2\vec{k}$

8. If $\vec{a}=\vec{i}-\vec{j}$, $\vec{b}=-+2\vec{i}+\vec{j}+\vec{k}$, $\vec{c}=\vec{i}+3\vec{j}+\vec{k}$, find $\vec{a}x\left(\vec{b}x\vec{c}\right)$

PART B

1. Find the Unit vector perpendicular to the following two vectors as well as the angle between them.

(i) $4\vec{i}+3\vec{j}+\vec{k}$ and $2\vec{i}-\vec{j}+2\vec{k}$

(ii) $-\vec{i}+\vec{j}+2\vec{k}$ and $-4\vec{i}+3\vec{j}-2\vec{k}$

(iii) $2\vec{i}+\vec{j}+\vec{k}$ and $\vec{i}+2\vec{j}+\vec{k}$

(iv) $3\vec{i}+\vec{j}+2\vec{k}$ and $\vec{i}+2\vec{j}+\vec{k}$

2. Show that the following points are Collinear

(i) $2\vec{i}+\vec{j}-\vec{k}$, $4\vec{i}+3\vec{j}-5\vec{k}$, $2\vec{i}+5\vec{j}-9\vec{k}$

(ii) $\vec{i}+2\vec{j}+4\vec{k}$, $4\vec{i}+8\vec{j}+\vec{k}$, $3\vec{i}+6\vec{j}+2\vec{k}$

(iii) $2\vec{i}-\vec{j}+3\vec{k}$, $3\vec{i}-5\vec{j}+\vec{k}$, $-\vec{i}+11\vec{j}+9\vec{k}$

3. Find the area of the triangle whose vertices are given as position vectors:

(i) $\vec{i}+2\vec{j}-\vec{k}$, $2\vec{i}+3\vec{k}$, $3\vec{i}-\vec{j}+2\vec{k}$

(ii) $\vec{i}+\vec{j}+\vec{k}$, $2\vec{i}+3\vec{j}-\vec{k}$, $3\vec{i}+\vec{k}$

(iii) (3,1,2),(1,-1,-3) and (4,-3,1)

(iv) (1,3,4),(-2,1,-1) and (0,-3,2)

4. Find the moment of the force $6\vec{i}+\vec{j}-\vec{k}$ acting along the point (0,1,-1) about the point (4,3,-1).

5. The force $\vec{i}+2+3\vec{k}$ is acting along the point $\vec{i}+\vec{j}+\vec{k}$ Find the moment of the force about the point $2\vec{i}-\vec{j}+\vec{k}$

6. A force $3\vec{i}+\vec{j}+2\vec{k}$ is acting along $\vec{i}-\vec{j}+2\vec{k}$ Find the moment of the force about the point
$\vec{a}=3\vec{i}+\vec{j}-\vec{k},\vec{b}=-\vec{i}+2\vec{j}-3\vec{k},\vec{c}=-\vec{i}-\vec{j}+3\vec{k},\vec{d}=\vec{i}+\vec{j}-\vec{k},$

7. Prove that the following four points line in a same plane

(i) $-6\vec{i}+3\vec{i}+2\vec{k},\ 3\vec{i}-2\vec{j}+4\vec{k},\ \ 5\vec{i}+7\vec{j}+3\vec{k},\ -13\vec{i}+17\vec{i}-\vec{k}$

(ii) (4,5,1),(0,-1,-1),(3,9,4),(-4,4,4)

(iii) (1,3,1),(1,1,-1),(-1,1,1),(2,2,-1)

(iv) (1,2,3),(3,-1,2),(-2,3,1),(6,-4,2)

8. If $\vec{a}=3\vec{i}+2\vec{j}-4\vec{k},\vec{b}=5\vec{i}-3\vec{j}+6\vec{k},\vec{c}=5\vec{i}-\vec{j}+2\vec{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$

9. If $\vec{a}=\vec{i}+\vec{j}+\vec{k},\vec{b}=3\vec{i}-2\vec{j}+\vec{k},\vec{c}=2\vec{i}-j-4\vec{k}$ find $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

10. If $\vec{a}=\vec{i}-2\vec{j}+3\vec{k},\vec{b}=2\vec{i}+3\vec{j},\vec{c}=3\vec{i}-\vec{k}$ find (i) $\vec{a}.(\vec{b}\times\vec{c})$

(ii) $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

11. If $\vec{a}=2\vec{i}+3\vec{j}-\vec{k},\vec{b}=\vec{j}+\vec{k},\vec{c}=\vec{i}+\vec{k},\vec{d}=\vec{i}+\vec{j}+\vec{k}$, find

(i) $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})$

12. If $\vec{a}=\vec{i}-\vec{j}+\vec{k},\vec{b}=2\vec{i}+3\vec{i}-5\vec{k},\vec{c}=2\vec{i}+\vec{j}-2\vec{k},\vec{d}=3\vec{i}-\vec{j}+4\vec{k},$

show that $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}.\vec{c} & \vec{a}.\vec{d}\\ \vec{b}.\vec{c} & \vec{b}.\vec{d}\end{vmatrix}$

13. If $\vec{a}=2\vec{i}-\vec{j}+\vec{k},\vec{b}=-\vec{i}-\vec{j}-\vec{k},\vec{c}=2\vec{i}+3\vec{j}-\vec{k},\vec{d}=\vec{i}+\vec{j}-\vec{k}$, find $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$.

14. If $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}, \vec{b} = -\vec{i} + 2\vec{j} - 3\vec{k}, \vec{c} = -\vec{i} - \vec{j} + 3\vec{k}, \vec{d} = \vec{i} + \vec{j} - \vec{k}$, find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

15. If $\vec{a} = 3\vec{i} + 2\vec{j} - 4\vec{k}, \vec{b} = 5\vec{i} - 3\vec{j} + 6\vec{k}, \vec{c} = 5\vec{i} - \vec{j} + 2\vec{k}, \vec{d} = 3\vec{j} - 4\vec{k}$, find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

16. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

ANSWERS

PART - A

1. (i) $4\vec{i} - 12\vec{j} - 7\vec{k}$ (ii) $4\vec{i} - \vec{j} - 3\vec{k}$

   (iii) $16\vec{i} + 9\vec{j} - \vec{k}$ (iv) $5\vec{i} - 4\vec{j} - 11\vec{k}$

4. (i) $\sqrt{166}$ (ii) $\sqrt{69}$ (iii) $\sqrt{90}$

5. (i) 12 (ii) -35 (iii) -15

6. (i) 8 (ii) 5 (iii) 26

8. $-5\vec{i} - 5\vec{j} - 3\vec{k}$

PART - B

1. (i) $\dfrac{7\vec{i} - 6\vec{j} - 10\vec{k}}{\sqrt{185}}, \sin\theta = \dfrac{\sqrt{185}}{\sqrt{234}}$

   (ii) $\dfrac{-8\vec{i} - 10\vec{j} + \vec{k}}{3\sqrt{165}}, \sin\theta = \dfrac{\sqrt{165}}{\sqrt{174}}$

   (iii) $\dfrac{\vec{i} - \vec{j} + \vec{k}}{\sqrt{3}}, \sin\theta = \dfrac{\sqrt{3}}{2}$

   (iv) $\dfrac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}, \sin\theta = \dfrac{\sqrt{3}}{\sqrt{84}}$

3. (i) $\frac{\sqrt{62}}{2}$, (ii) $\frac{\sqrt{62}}{2}$ (iii) $\frac{\sqrt{473}}{2}$ (iv) $\frac{\sqrt{933}}{2}$

4. $2\vec{i} - 4\vec{j} + 8\vec{k}, \sqrt{84}$

5. $6\vec{i} + 3\vec{j} - 4\vec{k}, \sqrt{61}$

6. $\vec{i} - \vec{j} - \vec{k}, \sqrt{3}$

8. $-95\vec{i} - 95\vec{j} + 190\vec{k}$

9. $-13\vec{i} + 8\vec{j} + 5\vec{k}$

10. $-34, 12\vec{i} - 4\vec{k}$

11. - 2

13. $2\vec{i} + 8\vec{j} + 4\vec{k}$

14. $-14\vec{i} - 14\vec{j} + 18\vec{k}$

15. $-190\vec{i} + 38\vec{j} - 76\vec{k}$

UNIT – III

INTEGRATION - I

Introduction:

3.1 Definition of integration – Integral values using reverse process of differentiation – Integration using decomposition method- Simple problems.

3.2 Integration by substitution: Integrals of the form $\int [f(x)]^n f'(x)dx$ where $n \neq -1$, $\int \frac{f'(x)}{f(x)} dx$,

$\int F[f(x)]\ f'(x)dx$ - Simple Problems .

3.3 Standard Integrals

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$, $\int \frac{dx}{\sqrt{a^2 - x^2}}$, $\int \frac{Ax + B}{ax^2 + bx + c} dx$

Simple Problems.

3.1 INTRODUCTION:

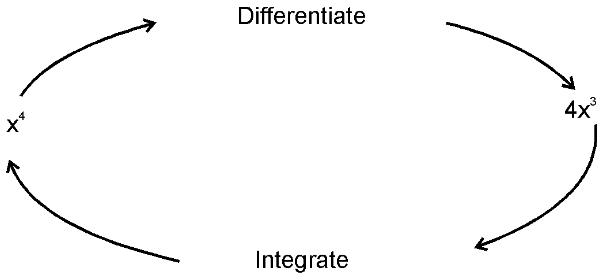
Sir Sardar Vallabhai Patel, called the Iron Man of India integrated several princely states together while forming our country Indian nation after independence. Like that in maths while finding area under a curve through integration, the area under the curve is divided into smaller rectangles and then integrating i.e., summing up all the areas of rectangles together. So integration means summation of very minute things of same kind.

Integration as the reverse of differentiation:

Integration can also be introduced in another way, called integration as the reverse of differentiation.

Differentiation in reverse:

Suppose we differentiate the function $y = x^4$. We obtain $\frac{dy}{dx} = 4x^3$ Integration reverse this process and we say that the integral of $4x^3$ as x^4. Pictorially we can think of this as follows:



The situation is just a little more complicated because there are lots of functions we can differentiate to give $4x^3$.

Here are some of them:

$$x^4 + 14, x^4 - 6, x^4 - 0.5, x^4 - \frac{1}{3}$$

Each of these functions has the same derivative $4x^3$, because when we differentiate the constant term we obtain zero. Consequently, when we try to reverse the process, we have no idea what the original constant term might have been. Because of this we include in our answer an unknown constant, C say, called the constant of integration. We state that the integral values of $4x^3$ is $x^4 + c$

The symbol for integration is $\int$, known as an integral sign. Formally we write

$$\int 4x^3 dx = x^4 + c$$

Along with the integral sign there is a term dx which must always be written and which indicates the name of the variable involved, in this case 'x'. Technically integrals of this sort are called Indefinite Integrals.

List of Formulae:

S. No	Differentiation	Integration
1	$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c\left(n \neq -1\right),$
2	$\frac{d}{dx}\left(\log x\right) = \frac{1}{x}$	$\int \frac{1}{x}dx = \log x + c$
3	$\frac{d}{dx}(e^x)=e^x$	$\int e^x dx = e^x + c$
4	$\frac{d}{dx}\left(\sin x\right)=\cos x$	$\int \cos x dx = \sin x + c$
5	$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + c$
6	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
7	$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x dx = -\cot x + c$
8	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + c$
9	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
10	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$ (or) $= -\cos^{-1} x + c$
11	$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$ (or) $= -\cot^{-1} x + c$
12	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + c$ (or) $= -\operatorname{cosec}^{-1} x + c$

Particular forms of $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ **where** $n \neq -1$

1. $\int x dx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$
2. $\int x^3 dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$
3. $\int dx = x + c$
4. $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$
5. $\int x^{-\frac{2}{3}} dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + c = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + c = 3x^{\frac{1}{3}} + c$

6 $\int \frac{1}{x^4} dx = \int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c = -\frac{x^{-3}}{3} + c$

Note that the final answer can be written in a variety of equivalent ways, for example

$$-\frac{1}{3}x^{-3} + c, \text{ or } \frac{1}{3}\ \frac{1}{x^3} + c \quad \text{or} \quad \frac{1}{3x^3} + c, \text{ or } -\frac{x^{-3}}{3} + c$$

Two Basic Theorems on Integration (without Proof)

1. If u, v, w etc are functions of x, then

   $\int (u \pm v \pm w \pm .....)\ dx = \int u dx \pm \int v dx \pm \int w dx \pm ....$

2. If f (x) is any function of x and k any constant then $\int kf(x)dx = k\int f(x)dx$

Example:

1. Evaluate $\int\left(7x^3 - \frac{2}{x^2} + \frac{4}{\sqrt{x}}\right)dx$

Solutions:

$$\int(7x^3 - \frac{2}{x^2} + \frac{4}{\sqrt{x}})\,dx = 7\int x^3dx - 2\int\frac{1}{x^2}dx + 4\int\frac{1}{\sqrt{x}}dx$$

$$= 7\int x^3dx - 2\int x^{-2}dx + 4\int x^{-\frac{1}{2}}\,dx$$

$$= \frac{7x^4}{4} - \frac{2x^{-2+1}}{-2+1} + \frac{4x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{7x^4}{4} - \frac{2x^{-1}}{-1} + \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{7x^4}{4} + 2x^{-1} + 8x^{\frac{1}{2}} + c$$

Integration using decomposition method:

In integration, there is no rule for multiplication or division of algebraic or trigonometric function as we have in differentiation Such functions are to be decomposed into addition and subtraction before applying integration.

For example: $\frac{\sin^2 x}{1+\cos x}$ can be decomposed as follows

$$\frac{\sin^2 x}{1+\cos x} = \frac{1-\cos^2 x}{1+\cos x} = \frac{(1+\cos x)(1-\cos x)}{(1+\cos x)}$$

= 1-cosx which can be integrated by using above basic theorems.

Examples:

1. **Evaluate** $\int\left(x^2+1\right)\left(x^2-\frac{1}{x^2}\right)dx$

Solution:

$$\int\left(x^2+1\right)\left(x^2-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^4-1+x^2-\frac{1}{x^2}\right)dx$$

$$=\int\left(x^4+x^2-x^{-2}-1\right)dx$$

$$=\frac{x^5}{5}+\frac{x^3}{3}-\frac{x^{-1}}{-1}-x+c$$

$$=\frac{x^5}{5}+\frac{x^3}{3}+x^{-1}-x+c$$

2. **Evaluate** $\int(1+x^2)^3dx$

Solution:

$$\int(1+x^2)^3dx=\int(1+3x^2+3x^4+x^6)dx$$

$$=x+\frac{3x^3}{3}+\frac{3x^5}{5}+\frac{x^7}{7}+c$$

$$=x+x^3+\frac{3x^5}{5}+\frac{x^7}{7}+c$$

3. Evaluate $\int\frac{\sin x}{1+\sin x}dx$

Solution:

$$\int\frac{\sin x}{1+\sin x}dx=\int\frac{\sin x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx$$

$$=\int\frac{\sin x-\sin^2 x}{1-\sin^2 x}dx$$

$$=\int\frac{\sin x-\sin^2 x}{\cos^2 x}dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int \left(\frac{\sin x}{\cos x} \frac{1}{\cos x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int (\tan x \sec x - \tan^2 x) dx$$

$$= \int [\tan x \sec x - (\sec^2 x - 1)] dx$$

$$= \int [\tan x \sec x - \sec^2 x + 1] dx$$

$$= \sec x - \tan x + x + c$$

Integrals of function containing linear expression in x (ie) ax+b

If $\int f(x)dx = g(x) + c$ then

$$\int f(ax+b)dx = \frac{1}{a} g(ax+b) + c$$

The extended forms of fundamental formulae:

1. $\int (ax+b)^n dx = \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + c, [n \neq -1]$

2. $\int \frac{1}{ax+b} dx = \frac{1}{a} \log(ax+b) + c$

3. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

4. $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + c$

5. $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + c$

6. $\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + c$

7. $\int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + c$

8. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+c$

9. $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\cot(ax+b)+c$

10. $\int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}\tan^{-1}(ax+b)+c$

11. $\int \frac{1}{\sqrt{1-(ax+b)^2}}dx = \frac{1}{a}\sin^{-1}(ax+b)+c$

Example:

1. Evaluate $\int (3-4x)^7 dx$

Solution:

$$\int (3-4x)^7 dx = \left(-\frac{1}{4}\right)\frac{(3-4x)^8}{8}+c$$

$$= -\frac{(3-4x)^8}{32}+c$$

2. Evaluate $\int e^{8-4x}dx$

Solution:

$$\int e^{8-4x}dx = \left(-\frac{1}{4}\right)e^{8-4x}+c$$

$$= -\frac{e^{8-4x}}{4}+c$$

3.1 WORKED EXAMPLES

PART - A

1. Evaluate $\int (2\sec^2 x + 5\cos x - \frac{4}{x} + 2e^x)dx$

Solution:

$$\int\left(2\sec^2 x + 5\cos x - \frac{4}{x} + 2e^x\right)dx$$

= 2tanx+5sinx-4logx+2ex+c

2. Evaluate $\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$

Solution:

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} + x^{\frac{-1}{2}}\right)dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{2x^{\frac{3}{2}}}{3} + 2x^{\frac{1}{2}} + c$$

3. Evaluate $\int (1+x^3)^2 dx$

Solution:

$$\int (1+x^3)^2 dx = \int (1+2x^3+x^6)dx$$

$$= x + \frac{2x^4}{4} + \frac{x^7}{7} + c$$

$$= x + \frac{x^4}{2} + \frac{x^7}{7} + c$$

4. Evaluate $\int (2x-5)(36+4x)dx$

Solution:

$$\int (2x-5)(36+4x)dx = \int (72x+8x^2-180-20x)dx$$

$$= \int (8x^2+52x-180)dx$$

$$=\frac{8x^3}{3}+\frac{52x^2}{2}-180x+c$$

$$=\frac{8x^3}{3}+26x^2-180x+c$$

5. Evaluate $\int \frac{x^3+4x^2-3x+2}{x^2}dx$

Solution:

$$\int \frac{x^3+4x^2-3x+2}{x^2}dx = \int (\frac{x^3}{x^2}+\frac{4x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2})dx$$

$$= \int (x+4-\frac{3}{x}+2x^{-2})dx$$

$$=\frac{x^2}{2}+4x-3\log x+\frac{2x^{-1}}{-1}+c$$

$$=\frac{x^2}{2}+4x-3\log x-2x^{-1}+c$$

6. Evaluate $\int \frac{1-x^2}{1-x}dx$

Solution:

$$\int \frac{1-x^2}{1-x}dx = \int \frac{(1+x)(1-x)}{(1-x)}dx$$

$$= \int (1+x)dx = x+\frac{x^2}{2}+c$$

7. Evaluate $\int \tan^2 x\,dx$

Solution:

$$\int \tan^2 x\,dx = \int (\sec^2 x-1)dx$$

$$= \tan x - x + c$$

8. Evaluate $\int(\sec x+\tan x)^2 dx$

Solution:

$$\int(\sec x+\tan x)^2 dx = \int(\sec^2 x+\tan^2 x+2\sec x\tan x)dx$$

$$= \int(\sec^2 x+\sec^2 x-1+2\sec x\tan x)dx$$

$$= \int(2\sec^2 x+2\sec x\tan x-1)dx$$

$$= 2\tan x+2\sec x-x+c$$

9. Evaluate $\int\sqrt{1-\sin 2x}\,dx$

Solution:

$$\int\sqrt{1-\sin 2x}\,dx = \int\sqrt{\sin^2 x+\cos^2 x-2\sin x\cos x}\,dx$$

$$= \int\sqrt{(\sin x-\cos x)^2}dx$$

$$= \int(\sin x-\cos x)dx$$

$$= -\cos x-\sin x+c$$

10. Evaluate $\int\frac{1}{1+\sin x}dx$

Solution:

$$\int\frac{1}{1+\sin x}dx = \int\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx$$

$$\int\frac{1-\sin x}{1-\sin^2 x}dx$$

$$= \int\frac{1-\sin x}{\cos^2 x}dx$$

$$= \int\left(\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int\left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\times\frac{1}{\cos x}\right)dx$$

$$= \int(\sec^2 x - \tan x \sec x)dx$$

$$= \tan x - \sec x + c$$

11. Evaluate $\int(4x+5)^6 dx$

Solution:

$$\int(4x+5)^6 dx = \left(\frac{1}{4}\right)\frac{(4x+5)^7}{7} + c$$

$$= \frac{(4x+5)^7}{28} + c$$

12. Evaluate $\int e^{3x+2}dx$

Solution:

$$\int e^{3x+2}dx = \left(\frac{1}{3}\right)e^{3x+2} + c$$

$$= \frac{e^{3x+2}}{3} + c$$

13. Evaluate $\int(3-2x)^3 dx$

Solution:

$$\int(3-2x)^3 dx = \left(-\frac{1}{2}\right)\frac{(3-2x)^4}{4} + c$$

$$= -\frac{(3-2x)^4}{8} + c$$

14. **Evaluate** $\int \sin(3x+1)dx$

Solution:

$$\int \sin(3x+1)dx = \left(-\frac{1}{3}\right)\cos(3x+1) + c$$

$$= -\frac{\cos(3x+1)}{3} + c$$

15. Evaluate $\int \frac{1}{\sqrt{1-2x}}dx$

Solution:

$$\int \frac{1}{\sqrt{1-2x}}dx = \int (1-2x)^{-\frac{1}{2}}dx$$

$$= \left(-\frac{1}{2}\right)\frac{(1-2x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$= -\frac{1}{2}\frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}}+c = -(1-2x)^{\frac{1}{2}}+c$$

PART – B

1. Integrate the following w.r.t x:

(a) $(x+\frac{1}{x})^3$

(b) $(1-x+x^2)(1+x+x^2)$

(c) $\frac{2}{x^2}-\frac{7}{x}+\frac{3}{\sin^2 x}$

(d) $(2x+3)^2(x-1)$

Solution:

(a) $\left(x+\frac{1}{x}\right)^3 = x^3+3x\frac{1}{x}\left(x+\frac{1}{x}\right)+\frac{1}{x^3}$

$$= x^3+3x+\frac{3}{x}+\frac{1}{x^3}$$

$$= x^3+3x+\frac{3}{x}+x^{-3}$$

$$\int\left(x+\frac{1}{x}\right)^3 = \int(x^3+3x+\frac{3}{x}+x^{-3})dx$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log x + \frac{x^{-2}}{-2} + c$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log x - \frac{x^{-2}}{2} + c$$

(b) $(1 - x + x^2)(1 + x + x^2)$

$$= (1 + x^2 - x)(1 + x^2 + x)$$

$$= (1 + x^2)^2 - x^2$$

$$= 1 + 2x^2 + x^4 - x^2$$

$$= 1 + x^2 + x^4$$

$$\int (1 - x + x^2)(1 + x + x^2)dx = \int (1 + x^2 + x^4)dx$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + c$$

(c) $\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x} \quad = 2x^{-2} - \frac{7}{x} + 3\operatorname{cosec}^2 x$

$$\int \left(\frac{2}{x^2} - \frac{7}{x} + \frac{3}{\sin^2 x} \right) dx = \int (2x^{-2} - \frac{7}{x} + 3\operatorname{cosec}^2 x)dx$$

$$= \frac{2x^{-1}}{-1} - 7\log x - 3\cot x + c$$

$$= -2x^{-1} - 7\log x - 3\cot x + c$$

(d) $(2x + 3)^2 (x - 1) \quad = (4x^2 + 12x + 9)(x - 1)$

$$= 4x^3 - 4x^2 + 12x^2 - 12x + 9x - 9$$

$$= 4x^3 + 8x^2 - 3x - 9$$

$$\int (2x + 3)^2 (x - 1)dx = \int (4x^3 + 8x^2 - 3x - 9)dx$$

$$= \frac{4x^4}{4} + \frac{8x^3}{3} - \frac{3x^2}{2} - 9x + c$$

2. Evaluate the following:

(a) $\int \frac{x^4 - x^2 + 1}{x + 1} dx$

(b) $\int \frac{1}{\sqrt{x+1} - \sqrt{x-2}} dx$

Solution:

(a) $$\frac{x^4 - x^2 + 1}{x + 1} = \frac{x^2(x^2 - 1) + 1}{x + 1}$$

$$= \frac{x^2(x^2 - 1)}{x + 1} + \frac{1}{x + 1}$$

$$= \frac{x^2(x + 1)(x - 1)}{x + 1} + \frac{1}{x + 1}$$

$$= x^2(x - 1) + \frac{1}{x + 1}$$

$$\int \frac{x^4 - x^2 + 1}{X + 1} dx = \int \left[x^2(x - 1) + \frac{1}{x + 1} \right] dx$$

$$= \int (x^3 - x^2 + \frac{1}{x + 1}) dx$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \log(x + 1) + c$$

(b)

$$\frac{1}{\sqrt{x+1} - \sqrt{x-2}} = \frac{1}{\sqrt{x+1} - \sqrt{x-2}} \times \frac{\sqrt{x+1} + \sqrt{x-2}}{\sqrt{x+1} + \sqrt{x-2}}$$

$$= \frac{\sqrt{x+1} + \sqrt{x-2}}{(\sqrt{x+1})^2 - (\sqrt{x-2})^2}$$

$$= \frac{(x+1)^{\frac{1}{2}} + (x-2)^{\frac{1}{2}}}{x + 1 - (x - 2)}$$

$$= \frac{(x+1)^{\frac{1}{2}} + (x-2)^{\frac{1}{2}}}{x + 1 - x + 2}$$

$$= \frac{(x+1)^{\frac{1}{2}} + (x-2)^{\frac{1}{2}}}{3}$$

$$\int \frac{1}{\sqrt{x+1}-\sqrt{x-2}}dx = \frac{1}{3}\int\left[(x+1)^{\frac{1}{2}}+(x-2)^{\frac{1}{2}}\right]dx$$

$$= \frac{1}{3}\left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}+\frac{(x-2)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$

$$= \frac{2}{9}\left[(x+1)^{\frac{3}{2}}+(x-2)\frac{3}{2}\right]+c$$

3. Evaluate the following:

(a) $\int \frac{\cos^2 x}{1+\sin x}dx$

(b) $\int \sin 3x \sin x \, dx$

(c) $\int \sin^3 x \, dx$

Solution:

(a) $\frac{\cos^2 x}{1+\sin x} = \frac{1-\sin^2 x}{1+\sin x}$

$$= \frac{(1+\sin x)(1-\sin x)}{(1+\sin x)}$$

$$= 1-\sin x$$

$$\int \frac{\cos^2 x}{1+\sin x}dx = \int (1-\sin x)dx$$

$$= x-(-\cos x)+c$$

$$= x+\cos x+c$$

(b) $\sin 3x \sin x = \frac{1}{2}[2\sin 3x \sin x]$

$[2\sin A \sin B = \cos(A-B)-\cos(A+B)]$

$$= \frac{1}{2}[\cos(3x - x) - \cos(3x + x)]$$

$$= \frac{1}{2}\left[\cos 2x - \cos 4x\right]$$

$$\int \sin 3x \sin x dx = \frac{1}{2} \int\left[\cos 2x - \cos 4x\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4}\right] + c$$

(c) $\sin 3A = 3\sin A - 4\sin^3 A$

$$4\sin^3 A = 3\sin A - \sin 3A$$

$$\sin^3 A = \frac{1}{4}\left[3\sin A - \sin 3A\right]$$

$$\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$$

$$\int \sin^3 x dx = \frac{1}{4} \int [3\sin x - \sin 3x] dx$$

$$= \frac{1}{4}[-3\cos x - \frac{(-\cos 3x)}{3}] + c$$

$$= \frac{1}{4}[-3\cos x + \frac{\cos 3x}{3}] + c$$

3.2 INTEGRATION BY SUBSTITUTION

So far we have dealt with functions, either directly integrable using integration formula or integrable after decomposing the given functions into sums or differences.

But there are functions like $\frac{\sin(\log x)}{x}, \frac{2x+3}{x^2+3x+5}$ etc which cannot be decomposed into sums or differences of simple functions.

In these cases, using proper substitution, we shall reduce the given form into standard form, which can be integrated using basic integration formula.

When the integrand (the function to be integrated) is either in multiplication or in division form and if the derivative of one full or meaningful part of the function is equal to the other function then the integration can be evaluated using substitution method as given in the following examples.

1 $\displaystyle\int \frac{2x+3}{x^2+3x+5}dx$

Since $\dfrac{d}{dx}(x^2+3x+5)$ is 2x+3 it can be integrated by taking $u = x^2+3x+5$

2. $\displaystyle\int \frac{\sin(\log x)}{x}dx = \int \sin(\log x)\frac{1}{x}dx$

Hare $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

The above integration can be evaluated by taking u = log x

Integrals of Some Standard Forms:

Integrals of the form $\int [f(x)]^n f'(x)dx$, $\int \frac{f'(x)}{f(x)}dx$, $\int f[f(x)]f'(x)dx$ are all, more or less of the same type and the use of substitution $u = f(x)$ will reduce the given function to simple standard form which can be integrated using integration formulae.

Examples:

1. Integrate $(x^2+7x-3)^{10}(2x+7)$ w.r.t x.

Solution:

Let I = $\int (x^2+7x-3)^{10}(2x+7)dx$

put $u = x^2+7x-3$

$du=(2x+7)dx$

$$I = \int (x^2+7x-3)^{10}(2x+7)dx$$

$$= \int u^{10}du = \frac{u^{11}}{11}+c$$

$$= \frac{\left(x^2+7x-3\right)^{11}}{11}+c$$

2. Evaluate $\int \frac{1+\cos x}{x+\sin x}dx$

Solution:

Put $u = x+\sin x$

$du = (1+\cos x)dx$

$$\int \frac{1+\cos x}{x+\sin x}dx = \int \frac{1}{u}du$$

$$= \log u + c$$

$$= \log(x+\sin x)+c$$

3. Evaluate $\int \sec^2(\sin x)\cos x\,dx$

Solution:

Now put $u = \sin x$

$du = \cos x\ dx$

$$\int \sec^2(\sin x)\cos x\ dx$$

$$= \int \sec^2 u\ du$$

$$= \tan u + c$$

$$= \tan(\sin x)+c$$

3.2 WORKED EXAMPLES
PART – A

1. Evaluate $\int \cot x \, dx$

Solution:

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

Put $u = \sin x$

$du = \cos x \, dx$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

$$\int \frac{1}{u} du = \log u + c$$

$$= \log \ (\sin x) + c$$

2. Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Solution:

Put $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}} dx \Rightarrow 2du = \frac{1}{\sqrt{x}} dx$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos \ u \ du$$

$$= 2\sin u + c$$

$$= 2\sin\sqrt{x} + c$$

3. Find the value of $\int \frac{(\log x)^5}{x} dx$

Solution:

Put $u = \log x$

$$du = \frac{1}{x} dx$$

$$\int \frac{(\log x)^5}{x} dx = \int u^5 du = \frac{u^6}{6} + c = \frac{(\log x)^6}{6} + c$$

PART - B

1. Find the value of the following:

a. $\int \sec x \; dx$

b. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

c. $\int x^2 \cos(x^3) dx$

d. $\int \frac{\sec^2 x}{(2+3\tan x)^3} dx$

e. $\int \tan x \sqrt{\sec x}\, dx$

Solution:

a. $\int \sec x \; dx \quad = \int \sec x \times \frac{\sec x + \tan x}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Put $u = \sec x + \tan x$

$du = (\sec x + \tan x + \sec^2 x) dx$

$$\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{1}{u} du = \log u + c$$

$$= \log(\sec x + \tan x) + c$$

b. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

Put $u = \sin^{-1} x$

$$du = \frac{1}{\sqrt{1-x^2}} dx$$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int u du$$

$$= \frac{u^2}{2} + c = \frac{(\sin^{-1} x)^2}{2} + c$$

c. $\int x^2 \cos(x^3) dx$

Put $u = x^3$

$du = 3x^2 dx$

$\frac{1}{3} du = x^2 dx$

$$\therefore \int x^2 \cos(x^3) dx = \frac{1}{3} \int \cos u \ du$$

$$= \frac{1}{3} \sin u + c = \frac{1}{3} \sin(x^3) + c$$

d. $\int \frac{\sec^2 x}{(2+3\tan x)^3} dx$

Put $= 2 + 3\tan x$

$du = 3\sec^2 x \, dx$

$\frac{1}{3} du = \sec^2 x \ dx$

$$\int \frac{\sec^2 x}{(2+3\tan x)^3} dx = \frac{1}{3} \int \frac{1}{u^3} du = \frac{1}{3} \int u^{-3} du$$

$$= \frac{1}{3} \frac{u^{-2}}{-2} + c = -\frac{(2+3\tan x)^{-2}}{-6} + c$$

e. $\int \tan x \ \sqrt{\sec x} \ dx$

$$= \int \frac{\tan x \sqrt{\sec x} \times \sqrt{\sec x}}{\sqrt{\sec x}} dx$$

$$= \int \frac{\tan x \sec x}{\sqrt{\sec x}} dx$$

Put $\quad u = \sec x$

$$du = \sec x \tan x \ dx$$

$$\int \tan x \sqrt{\sec x} \ dx = \int \frac{1}{\sqrt{u}} du$$

$$= \int u^{-\frac{1}{2}} du$$

$$= \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{\sec x} + c$$

2. Evaluate the following:

a $\int (2x^2 - 8x + 5)^{11} (x-2) dx$

b. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

c. $\int e^{\sin^2 x} \sin 2x \, dx$

d. $\int \frac{\cos x}{a + b \sin x} dx$

(a) Put $u = 2x^2 - 8x + 5$

$$du = (4x-8)dx = 4(x-2)dx \qquad \frac{du}{4} = (x-2)dx$$

$$\int (2x^2 - 8x + 5)^{11} (x-2) dx$$

$$= \int u^{11} \frac{du}{4} = \frac{1}{4} \int u^{11} du$$

$$= \frac{1}{4}\frac{u^{12}}{12} + c = \frac{(2x^2 - 8x + 5)^{12}}{48} + c$$

(b) Put $u = e^x - e^{-x}$

$du = [e^x-(-e^{-x})]\, dx$

$= (e^x+e^{-x})\, dx$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{1}{u} du$$

$$= \log u + c = \log(e^x - e^{-x}) + c$$

(c) Put $u = \sin^2 x$

$du = 2\sin x \cos x\, dx$

$= \sin 2x\, dx$

$$\int e^{\sin^2 x}.\sin 2x \; dx = \int e^u du = e^u + c = e^{\sin^2 x} + c$$

(d) Put $u = a+b \sin x$ $du = b \cos x\, dx$

$$\frac{1}{b} du = \cos x\, dx$$

$$\int \frac{\cos x}{a + b\sin x} dx = \frac{1}{b}\int \frac{1}{u} du = \frac{1}{b}\log u + c = \frac{1}{b}\log(a + b\sin x) + c$$

3.3 STANDARD INTEGRALS

Integrals of the form $\int\frac{dx}{a^2 \pm x^2}$, $\int\frac{dx}{x^2+a^2}$ and $\int\frac{dx}{\sqrt{a^2-x^2}}$

1. $$\int\frac{dx}{a^2+x^2} = \frac{1}{a^2}\int\frac{dx}{1+\left(\frac{x}{a}\right)^2}$$

$$= \frac{1}{a^2}\int\frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right)\times a$$

$$= \frac{1}{a}\int\frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

2. $$\int\frac{dx}{a^2-x^2} = \int\frac{1}{(a+x)(a-x)}dx$$

$$= \frac{1}{2a}\int\frac{2a}{(a+x)(a-x)}dx$$

$$= \frac{1}{2a}\int\frac{(a+x)+(a-x)}{(a+x)(a-x)}dx$$

$$= \frac{1}{2a}\int\left[\frac{1}{a-x}+\frac{1}{a+x}\right]dx$$

$$= \frac{1}{2a}[-\log(a-x)+\log(a+x)]+c$$

$$= \frac{1}{2a}\log\frac{a+x}{a-x}+c$$

3. $\int \frac{1}{x^2 - a^2} dx \quad = \quad \frac{1}{2a} \int \frac{2a}{(x+a)(x-a)} dx$

$$= \frac{1}{2a} \int \frac{(x+a)-(x-a)}{(x+a)(x-a)} dx$$

$$= \frac{1}{2a} \int [\frac{1}{x-a} - \frac{1}{x+a}] dx$$

$$= \frac{1}{2a} [\log(x-a) - \log(x+a)] + c$$

$$= \frac{1}{2a} \log \frac{x-a}{x+a} + c$$

4. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx$

$$= \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right) \times a$$

$$= \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right) =$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c$$

So remember,

1. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

2. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

3. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

4. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$

3.3 WORKED EXAMPLES

PART - A

1. Evaluate $\int \frac{dx}{9+x^2}$

Solution:

We known that $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})+c$

$$\int \frac{dx}{9+x^2} = \int \frac{dx}{3^2+x^2}$$

$$= \frac{1}{3}\tan^{-1}(\frac{x}{3})+c$$

2. Evaluate $\int \frac{1}{7-x^2}dx$

Solution:

We known that $\int \frac{dx}{a^2-x^2}dx = \frac{1}{2a}\log\frac{a+x}{a-x}+c$

$$\int \frac{1}{7-x^2}dx = \int \frac{1}{(\sqrt{7})^2-x^2}dx$$

$$= \frac{1}{2\sqrt{7}}\log\frac{\sqrt{7}+x}{\sqrt{7}-x}+c$$

3. Evaluate $\int \frac{dt}{\sqrt{25-t^2}}$

Solution:

We known that $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$

$$\int \frac{dt}{\sqrt{25-t^2}} = \int \frac{dt}{\sqrt{5^2-t^2}} = \sin^{-1}\left(\frac{t}{5}\right)+c$$

4. Evaluate $\int \frac{dt}{t^2 - 16}$

Solution:

We known that $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

$$\int \frac{dt}{t^2 - 16} = \int \frac{dt}{t^2 - 4^2} = \frac{1}{2(4)} \log \frac{t-4}{t+4} + c$$

$$= \frac{1}{8} \log \frac{t-4}{t+4} + c$$

5. Evaluate $\int \frac{dx}{2 - 3x^2}$

Solution:

We known that $\int \frac{dx}{a^2 - x^2} \quad = \frac{1}{2a} \log \frac{a+x}{a-x} + c$

$$\int \frac{dx}{2 - 3x^2} = \frac{1}{3} \int \frac{dx}{\frac{2}{3} - x^2} = \frac{1}{3} \int \frac{dx}{(\sqrt{\frac{2}{3}})^2 - x^2}$$

$$= \frac{1}{3} \times \frac{1}{2 \times \sqrt{\frac{2}{3}}} \log \frac{\sqrt{\frac{2}{3}} + x}{\sqrt{\frac{2}{3}} - x} + c$$

$$= \frac{\sqrt{3}}{3 \times 2 \times \sqrt{2}} \log \frac{\sqrt{2} + \sqrt{3}x / \sqrt{3}}{\sqrt{2} - \sqrt{3}x / \sqrt{3}} + c$$

$$= \frac{1}{2\sqrt{6}} \log \frac{\sqrt{2} + \sqrt{3}x}{\sqrt{2} - \sqrt{3}x} + c$$

PART - B

1. Evaluate $\int \frac{dx}{(3x+2)^2 - 16}$

Solution:

$$\int \frac{dx}{(3x+2)^2 - 16}$$ take $t = 3x+2$

$$= \frac{1}{3} \int \frac{dt}{t^2 - 4^2}$$ $dt = 3dx$

$\frac{dt}{3} = dx$

$$= \frac{1}{3} \times \frac{1}{2(4)} \log \frac{t-4}{t+4} + c$$

$$= \frac{1}{24} \log \frac{3x+2-4}{3x+2+4} + c$$

$$= \frac{1}{24} \log \frac{3x-2}{3x+6} + c$$

2. Find the value of $\int \frac{1}{(3x+2)^2 + 16} + dx$

Solution:

$$\int \frac{1}{(3x+2)^2 + 16} dx$$ take $u = 3x+2$

$$= \frac{1}{3} \int \frac{1}{u^2 + 4^2} du$$ $du = 3dx$

$\frac{du}{3} = dx$

$$= \frac{1}{3} \times \frac{1}{4} \tan^{-1} \frac{u}{4} + c$$

$$= \frac{1}{12} \tan^{-1}\left(\frac{3x+2}{4}\right) + c$$

3. Find the value of $\int \frac{dx}{49-9x^2}$

Solution:

$$\int \frac{dx}{49-9x^2} = \int \frac{dx}{7^2-(3x)^2} \qquad \text{take } t = 3x$$

$$= \frac{1}{3}\int \frac{dt}{7^2-t^2} \qquad dt = 3dx$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}+c \qquad \frac{dt}{3} = dx$$

$$\frac{1}{3}\times\frac{1}{2(7)}\log\frac{7+t}{7-t}+c$$

$$\frac{1}{42}\log\frac{7+3x}{7-3x}+c$$

4. Evaluate $\int \frac{1}{\sqrt{121-12x^2}}dx$

Solution:

We known that $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c$

$$\int \frac{1}{\sqrt{121-12x^2}}dx = \int \frac{1}{\sqrt{(11)^2-(2\sqrt{3}x)^2}}dx$$

$$= \int \frac{\frac{dt}{2\sqrt{3}}}{\sqrt{(11)^2-t^2}} \qquad \text{taking } t = 2\sqrt{3}x$$

$$= \frac{1}{2\sqrt{3}}\int \frac{dt}{\sqrt{(11)^2-t^2}} \qquad dt = 2\sqrt{3}\,dx$$

$$= \frac{1}{2\sqrt{3}}\sin^{-1}\left(\frac{t}{11}\right)+c \qquad \frac{dt}{2\sqrt{3}} = dx$$

$$= \frac{1}{2\sqrt{3}}\sin^{-1}\left(\frac{2\sqrt{3}x}{11}\right)+c$$

Method of partial fraction

When we integrate algebraic rational functions in the form $\frac{p(x)}{q(x)}$, the degree of p(x) may be greater or less then that of q(x). If it is greater or equal degree, first we shall divide p(x) by q(x) and if $\frac{p(x)}{q(x)} = Q + \frac{p_1(x)}{q(x)}$, when Q is the quotient and $p_1(x)$, is the reminder and is of degree less then q(x).

If the denominator q(x) can be factorized into two or more factors, then we can express $\frac{p_1(x)}{q(x)}$ as algebraic sum of two or more fractions, called Partial fractions. Then integrating the rational function becomes simpler and we can integrate term by term, the partial fractions.

Example:

1. Evaluate $\int \frac{2x^4 + x^3 - 5x^2 + 5x + 3}{x^2 + x - 2} dx$

Solution:

$$
\begin{array}{r}
x^2 + x - 2)\,2x^4 + x^3 - 5x^2 + 5x + 3\,(2x^2 - x \\
2x^4 + 2x^3 - 4x^2 \qquad\qquad\quad \\
\hline
-x^3 - x^2 + 5x^2 \qquad \\
-x^3 - x^2 + 2x \qquad\quad \\
\hline
3x + 3 \qquad
\end{array}
$$

$$\frac{2x^4 + x^3 - 5x^2 + 5x + 3}{x^2 + x - 2} = 2x^2 - x + \frac{3x + 3}{x^2 + x - 2}$$

$$= 2x^2 - x + \frac{3x + 3}{(x - 1)(x + 2)}$$

Now let $\frac{3x + 3}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$

$$= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$3x + 3 = A(x+2) + B(x-1)$$

When $x=1$, $3(1) + 3 = A(1+2) + 0$

$$6 = 3A \Rightarrow A = 2$$

When $x = -2$, $3(-2) + 3 = A(-2+2) + B(-2-1)$

$$-6+3 = 0-3B$$

$$-3 = -3B \Rightarrow B = 1$$

Now $\int \frac{2x^4 + x^3 - 5x^2 + 5x + 3}{x^2 + x - 2} dx = \int (2x^2 - x + \frac{2}{x-1} + \frac{1}{x+2}) dx$

$$= \frac{2x^3}{3} - \frac{x^2}{2} + 2\log(x-1) + \log(x+2) + c$$

Note: If the function to be integrated is in the term $\frac{Ax+B}{lx^2+mx+n}$ and the denominator is not factorisable, then the following method is adopted.

Put,

Numerator = k_1 (exact derivative of Denominator) + k_2, where k_1 and k_2 are constants.

Then the function to be integrated will be divided into two parts and each can be integrated separately.

Example :

1. Find the integral $\int \frac{x+1}{x^2+6x+25} dx$

Solution:

Here the denominator is not factorisable.

Put,

Numerator = k_1 (exact derivative of denominator) $+k_2$

$$x+1 = k_1(2x+6) + k_2$$

Comparing coefficients of x terms, we have

$$1 = 2k_1 \Rightarrow k_1 = \frac{1}{2}{}_2$$

Comparing constant terms, we get

$$1 = 6k_1 + k_2$$

$$= 6\left(\frac{1}{2}\right) + k_2$$

$$1 = 3 + k_2 \Rightarrow k_2 = -2$$

$$\int \frac{x+1}{x^2+6x+25}dx = \int \frac{\frac{1}{2}(2x+6)+(-2)}{x^2+6x+25}dx$$

$$= \frac{1}{2}\int \frac{2x+6}{x^2+6x+25}dx - 2\int \frac{1}{x^2+6x+25}dx$$

$$= \frac{1}{2}\int \frac{d(x^2+6x+25)}{x^2+6x+25} - 2\int \frac{1}{(x+3)^2-9+25}dx$$

(by the method of making a perfect square)

$$= \frac{1}{2}\int \frac{du}{u} - 2\int \frac{1}{(x+3)^2+4^2}dx$$

taking $u = x^2 + 6x + 5$

$du = (2x+6)\ dx$

$$= \frac{1}{2}\int \frac{du}{u} - 2\int \frac{dt}{t^2+4^2} \qquad \text{taking } t = x+3$$

$dt = dx$

$$= \frac{1}{2}\log u - 2\times\frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right) + c$$

$$= \frac{1}{2}\log(x^2+6x+25) - \frac{1}{2}\tan^{-1}\left(\frac{x+3}{4}\right) + c$$

WORKED EXAMPLES

PART - B

1. Evaluate $\int \frac{dx}{x^2 - 3x + 2}$

Solution:

Let $\frac{1}{x^2 - 3x + 2} = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$1 = A\ (x-2) + B\ (x-1)$

When x=1; $1 = A\ (1-2) + B\ (1-1)$

$1 = A\ (-1) + 0$

$1 = -A \Rightarrow A = -1$

When x=2; $1 = A\ (2-2) + B\ (2-1)$

$1 = 0 + B(1)$

$1 = B$

$$\therefore \int \frac{dx}{x^2 - 3x + 2} = \int \left(\frac{-1}{x-1} + \frac{1}{x-2} \right) dx$$

$$= -\log(x-1) + \log(x-2) + c$$

$$= \log \frac{x-2}{x-1} + c$$

2. Evaluate $\int \frac{x+7}{x^2 + 2x - 8} dx$

Solution:

Let $\frac{x+7}{x^2 + 2x - 8} = \frac{x+7}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$

$\therefore x+7 = A\ (x-2) + B\ (x+4)$

When x=2; $2+7 = A\ (2-2) + B\ (2+4)$

$$9 \quad = 0 + 6B \Rightarrow B = \frac{9}{6} = \frac{3}{2}$$

When x= -4; -4+7= A (-4-2) + B (-4+4)

$$3 \quad = -6A+0 \Rightarrow A = -\frac{3}{-6} = -\frac{1}{2}$$

$$\therefore \int \frac{x+7}{x^2+2x-8} dx \quad = \int \left(\frac{-\frac{1}{2}}{x+4} + \frac{\frac{3}{2}}{x-2} \right) dx$$

$$= -\frac{1}{2}\log(x+4) + \frac{3}{2}\log(x-2) + c$$

3. Evaluate $\int \frac{3x+2}{x^2+x+1}$

Solution:

Here the denominator is not factorisable

$$\therefore 3x+2 \quad = A\left[\frac{d}{dx}(x^2+x+1)\right] + B$$

$$3x+2 \quad = A\ (2x+1) + B$$

Comparing coefficient of x;

$$3 = 2A \Rightarrow A = \frac{3}{2}$$

Comparing constant terms;

$$2. = A+B$$

$$= \frac{3}{2} + B$$

$$2 - \frac{3}{2} = B$$

$$\frac{4-3}{2} = B \Rightarrow \frac{1}{2}$$

$$\therefore \int \frac{3x+2}{x^2+2x+1}dx = \int \frac{\frac{3}{2}(2x+1)+\frac{1}{2}}{x^2+x+1}dx$$

$$= \frac{3}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{3}{2}\int \frac{du}{u} + \frac{1}{2}I \text{ where } u = x^2 + x + 1$$

$$= \frac{3}{2}\log u + \frac{1}{2}I \qquad \therefore du = (2x+1)dx$$

$$= \frac{3}{2}\log(x^2+x+1) + \frac{1}{2}I \qquad \text{and } I = \int \frac{1}{x^2+x+1}dx$$

$$\text{Now } I = \int \frac{1}{x^2+x+1}dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1}dx$$

(by making a perfect square)

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$I = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\frac{2x+1}{2}}{\frac{\sqrt{3}}{2}} \right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}} \right)$$

$$\therefore \int \frac{3x+2}{x^2+x+1} dx = \frac{3}{2} \log (x^2+x+1) + \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}} \right)$$

$$= \frac{3}{2} \log (x^2+x+1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$$

EXERCISE

PART –A

1. Evaluate the following
   - i $\int \frac{1}{1-\cos x} dx$
   - ii $\int \sec x(\sec x + \tan x)\, dx$
   - iii $\int \left(\sin\frac{x}{2} + \cos\frac{x}{2} \right)^2 dx$
   - iv $\int \operatorname{cosec}^2 (3x+4) dx$
   - v $\int (3x+4)^6 dx$
2. Evaluate the following
   - i $\int \sqrt{3x+5}\, dx$
   - ii $\int e^{9x-2} dx$
   - iii $\int \left(t + \frac{1}{t} \right)^2 dt$
   - iv $\int \sqrt{1+\cos 2x}\, dx$

v $\int \sqrt{1+\sin 2x}\,dx$

vi $\int \frac{1}{(4-5x)^7}dx$

vii $\int \sec(3x+4)\tan(3x+4)dx$

viii $\int \frac{1}{\cos^2(px+c)}dx$

ix $\int \frac{1}{\sin^2(l-mx)}dx$

x $\int \left(\frac{3}{x} - 2e^x + \operatorname{cosec}^2 x - 7\right)dx$

xi $\int \frac{ax^3+bx^2+cx+d}{x^2}dx$

3. Evaluate the following

i $\int \frac{6x}{3x^2-1}dx$

ii $\int \frac{\log x}{x}dx$

iii $\int e^{x^2}\,2x\,dx$

iv $\int \sqrt{\tan x}\,\sec^2 x\,dx$

v $\int \cot x\sqrt{\operatorname{cosec} x}\,dx$

vi $\int \frac{e^{\tan x}}{\cos^2 x}dx$

vii $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

viii $\int \frac{\cos x}{1+\sin x}dx$

ix $\int \frac{\sin x}{1-\cos x}dx$

x $\int \tan x dx$

xi $\int \operatorname{cosec} x dx$

4. Evaluate the following

i $\int \frac{1}{x^2 - 36} dx$

ii $\int \frac{dx}{9 - x^2}$

iii $\int \frac{1}{3 + 4x^2} dx$

iv $\int \frac{1}{\sqrt{2 - x^2}} dx$

v $\int \frac{1}{121 + t^2} dx$

vi $\int \frac{1}{4 - x^2} dx$

vii $\int \frac{1}{1 + \frac{x^2}{4}} dx$

viii $\int \frac{dx}{\sqrt{a^2 - b^2 x^2}}$

PART - B

1. Evaluate the following:

i $\int \left(x^2 - 1\right)\left(2 + x\right)^2 dx$

ii $\int (x - 1)(x + 2)(x + 1) dx$

iii $\int (x + \frac{1}{x})(\sqrt{x} - \frac{1}{x} dx$

iv $\int (x + \sqrt{x})^2 dx$

v $\int \frac{1-x^3}{1-x} dx$

vi $\int \frac{\left(2x^2+1\right)\left(3x^2-4x+5\right)}{x^2} dx$

vii $\int (1+x^2)(1-x)^2 dx$

viii $\int \frac{dx}{\sqrt{x+3}-\sqrt{x-4}}$

ix $\int (x-1)\sqrt{x+1}\ dx$

x $\int (3x+4)\sqrt{3x+7}dx$

xi $\int \frac{1}{\sqrt{ax+b}-\sqrt{ax+c}}\ dx$

2. Evaluate the Following

i $\int \frac{\sin x}{1-\sin x} dx$

ii $\int \frac{\cos x}{1+\cos x} dx$

iii $\int \frac{\cos x}{1-\cos x} dx$

iv $\int \frac{\sin^2 x}{1+\cos x} dx$

v $\int \frac{\sin^2 x}{1-\cos x} dx$

vi $\int \frac{\cos^2 x}{1-\sin x} dx$

vii $\int \cos^3 x dx$

viii $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

ix $\int \sin 5x \cos 2x dx$

x $\int \sin 7x \cos 5x dx$

xi $\int \cos 3x \cos x dx$

xii $\int \cos 2x \sin 4x dx$

xiii $\int \sin 10x \sin 2x dx$

xiv $\int \frac{1+\cos 2x}{\sin^2 x} dx$

xv $\int \frac{1-\cos 2x}{\cos^2 x} dx$

xvi $\int \frac{a + b\sin x}{\cos^2 x}$

3. Evaluate the following:

i $\int \sin^2 x \cos x dx$

ii $\int \sin x \cos^3 x dx$

iii $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

iv $\int x^2 \sqrt{1-x^3}\ dx$

v $\int \frac{1}{1+e^{-x}} dx$

vi $\int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x}} dx$

vii $\int \frac{(1+\log x)^{11}}{x} dx$

viii $\int \frac{\left(3+\sqrt{x}\right)^3}{\sqrt{x}} dx$

ix $\int \frac{\cot x}{\log(\sin x)} dx$

x $\int \frac{6x^2-1}{2x^2-x+5)} dx$

4. Integrate the following

i $(3x^2-6x+1)^{10}(x-1)$

ii $(2e^x-3)^{11}e^x$

iii $\frac{1}{x(3+\log x)^5}$

iv $\frac{\cos x}{\sin x+5}$

v $\frac{\sec^2 x}{3+\tan x}$

vi $\frac{e^x-\sin x}{e^x+\cos x}$

vii $\text{Sin}^7x \cos x$

viii $\text{Cos}^5x \sin x$

ix \\$e^{x\log x}(1+\log x)$

5. Evaluate the following

i. $\int \frac{1}{(2-x)^2+16} dx$

ii $\int \frac{1}{\left(3x+2\right)^2+16} dx$

iii $\int \frac{1}{\left(x-2\right)^2+9} dx$

iv $\int \frac{1}{(2x+3)^2+9}dx$

v $\int \frac{1}{9x^2-4}dx$

vi $\int \frac{1}{ax^2-b^2}dx$

vii $\int \frac{1}{(x+1)^2-9}dx$

viii $\int \frac{1}{(2x+3)^2-25}dx$

ix $\int \frac{1}{\sqrt{4-9x^2}}dx$

x $\int \frac{1}{\sqrt{4-(x+3)^2}}dx$

xi $\int \frac{1}{\sqrt{16-(x+1)^2}}dx$

xii $\int \frac{1}{\sqrt{25-(x+3)^2}}dx$

xiii $\int \frac{1}{4-2x^2}dx$

xiv $\int \frac{1}{25-(x+1)^2}dx$

xv $\int \frac{1}{36-(2x+3)^2}dx$

xvi $\int \frac{1}{16-(x-1)^2}dx$

6. Evaluate the following

i. $\int \frac{1}{(3-x)(x-5)}dx$

ii $\int \frac{1}{x(x-1)} dx$

iii $\int \frac{1}{2x^2 - x - 1} dx$

iv $\int \frac{x+3}{x(x+2)} dx$

ANSWERS
PART - A

1. (i) $-\cot x - \operatorname{cosec} x + c$

   (ii) $\tan x + \sec x + c$

   (iii) $x - \cos x + c$

   (iv) $-\frac{1}{3}\cot(3x+4) + c$

   (v) $\frac{(3x+4)^7}{21} + c$

2. (i) $\frac{2(3x+5)^{\frac{3}{2}}}{9} + c$

   (ii) $\frac{e^{9x-2}}{9} + c$

   (iii) $\frac{t^3}{3} - \frac{1}{t} + 2t + c$

   (iv) $\sqrt{2}\sin x + c$

   (v) $-\cos x + \sin x + c$

   (vi) $\frac{(4-5x)^{-6}}{30} + c$

(vii) $\dfrac{\sec(3x+4)}{3}+c$

(viii) $\dfrac{\tan(px+c)}{p}+c$

(ix) $\dfrac{\cot(l-mx)}{m}+c$

(x) $3\log x-2e^{x}-\cot x-7x+c$

(xi) $\dfrac{ax^2}{2}+bx+c\log x-\dfrac{dx^{-3}}{3}+c$

3. (i) $\log(3x^2-1)+c$ (ii) $\dfrac{(\log x)^2}{2}+c$ (iii) $e^{x^2}+c$

(iv) $\dfrac{2}{3}(\tan x)^{\frac{3}{2}}+c$ (v) $-2\sqrt{\operatorname{cosec} x}+c$ (vi) $e^{\tan x}+c$

(vii) $2e^{\sqrt{x}}+c$ (viii) $\log(1+\sin x)+c$ (ix) $\log(1-\cos x)+c$

(x) $\log \sec x+c$ (xi) $\log(\operatorname{cosec} x-\cot x)+c$

4. (i) $\dfrac{1}{12}\log\dfrac{x-6}{x+6}+c$ (ii) $\dfrac{1}{16}\log\dfrac{3+x}{3-x}+c$ (iii) $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2x}{\sqrt{3}}+c$

(iv) $\sin^{-1}\dfrac{x}{\sqrt{2}}+c$ (v) $\dfrac{1}{11}\tan^{-1}\dfrac{t}{11}+c$ (vi) $\sin^{-1}\dfrac{x}{2}+c$

(vii) $4\tan^{-1}\dfrac{x}{2}+c$ (viii) $\dfrac{1}{6}\sin^{-1}\dfrac{bx}{a}+c$

PART –B

1. (i) $\dfrac{x^5}{5}+x^4+x^3-2x^2-4x+c$

(ii) $\dfrac{x^4}{4}+\dfrac{2x^3}{3}-\dfrac{x^2}{2}-2x+c$

(iii) $\dfrac{2}{5}x^{\frac{5}{2}}-\dfrac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+2x^{-\frac{1}{2}}+c$

(iv) $\frac{x^3}{3}+\frac{x^2}{2}+\frac{4}{5}x^{\frac{5}{2}}+c$

(v) $x+\frac{x^2}{x}+\frac{x^3}{3}+c$

(vi) $2x^3-4x^2+13x-4\log x-\frac{5}{x}+c$

(vii) $\frac{x^5}{5}-\frac{x^4}{2}+\frac{2}{3}x^3-x^2x+c$

(viii) $\frac{2}{21}\left[(x+3)^{\frac{3}{2}}-(x-1)^{\frac{3}{2}}\right]+c$

(ix) $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+c$

(x) $\frac{2}{15}(3x+7)^{\frac{5}{2}}-2(3x+7)^{\frac{3}{2}}+c$

(xi) $\frac{2}{3a(b-c)}(ax+b)^{\frac{3}{2}}-\frac{2}{3a(b-c)}(ax+b)^{\frac{3}{2}+c}$

2. (i) secx+tanx-x+c

(ii)-cosecx +cotx+x+c

(iii)-cosecx-cotx-x-c

(iv)x-sinx+c

(v) x+sinx+c

(vi)x-cosx+c

(vii) $\frac{3}{4}\sin x+\frac{1}{12}\sin x+c$

(viii) tanx+cotx+c

(ix) $-\frac{1}{2}\left[\frac{\cos 7x}{7}+\frac{\cos 3x}{3}\right]+c$

(x) $-\frac{1}{2}\left[\frac{\cos 12x}{12}+\frac{\cos 2x}{2}\right]+c$

(xi) $\frac{1}{2}\left[\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right]+c$

(xii) $-\frac{1}{2}\left[-\frac{\cos 6x}{6}-\frac{\cos 2x}{2}\right]+c$

(xiii) $\frac{1}{2}\left[\frac{\sin 8x}{8}-\frac{\sin 12x}{2}\right]+c$

(xiv) $2[-\cot x-x]+c$

(xv) $2[\tan x-x]+c$

(xvi) a tanx + b secx + c

3. (i) $\frac{\sin^3 x}{3}+c$

(ii) $-\frac{\cos^4 x}{4}+c$

(iii) $e^{\tan-1}x+c$

(iv) $-\frac{2}{9}(1-x^3)^{\frac{3}{2}}+c$

(v) $\log(1+e^x)+c$

(vi) $-2\sqrt{\cot x}+c$

(vii) $\frac{1}{12}(1+\log x)^{12}+c$

(viii) $\frac{(3+\sqrt{x})^4}{2}+c$

(ix) log [log(sinx)]+c

(x) log $(2x^3-x+5)+5$

4. (i) $\frac{1}{66}(3x^2-6x+1)^{11}+c$

(ii) $\frac{1}{24}(2e^x-3)^{10}+c$

(iii) $\frac{1}{4(3+\log x)^4}+c$

(iv) log (sinx+5)+c

(v) log (3+tanx)+c

(vi) log $(e^x$+cosx)+c

(vii) $\frac{1}{8}\sin^8 x+c$

(viii) $-\frac{\cos^6 x}{6}+c$

(ix) $e^{x\log x}+c$

5. (i) $-\frac{1}{4}\tan^{-1}\left(\frac{2-x}{4}\right)+c$

(ii) $\frac{1}{12}\tan^{-1}\left(\frac{3x+2}{4}\right)+c$

(iii) $\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$

(iv) $\frac{1}{6}\tan^{-1}\left(\frac{2x+3}{3}\right)+c$

(v) $\frac{1}{12}\log\frac{3x-2}{3x+2}+c$

(vi) $\frac{1}{2\sqrt{ab}}\log\frac{\sqrt{a}x-b}{\sqrt{a}x+b}+c$

(vii) $\frac{1}{6}\log\left(\frac{x-2}{x+2}\right)+c$

(viii) $\frac{1}{20}\log\left(\frac{x-1}{x+4}\right)+c$

(ix) $\frac{1}{3}\sin^{-1}\frac{3x}{2}+c$

(x) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(xi) $\sin^{-1}\left(\frac{x+1}{4}\right)+c$

(xii) $\sin^{-1}\left(\frac{x+3}{5}\right)+c$

(xiii) $\frac{1}{4\sqrt{2}}\log\left(\frac{2+\sqrt{2}x}{2-\sqrt{2}x}\right)+c$

(xiv) $\frac{1}{10}\log\left(\frac{6+x}{4-x}\right)+c$

(xv) $\frac{1}{24}\log\left(\frac{9+2x}{3-2x}\right)+c$

(xvi) $\frac{1}{8}\log\left(\frac{3+x}{5-x}\right)+c$

6. (i) $\frac{1}{2}\log\frac{x-5}{x-3}+c$

(ii) $\log\left(\frac{x-1}{x}\right)+c$

(iii) $\frac{1}{3}\log(x-1)-\frac{2}{3}\log(2x+1)+c$

(iv) $\frac{3}{2}\log x-\frac{1}{2}\log(x+2)+c$

UNIT – IV

INTEGRATION - II

4.1 INTEGRATION BY PARTS

Integrals of the form $\int x \sin nx\, dx$, $\int x \cos nx\, dx$, $\int xe^{nx}dx$, $\int x^n \log x dx$, $\int x \log x\, dx$ - Simple Problems

4.2 BERNOULLI'S FORMULA

Evaluation of the integrals $\int x^m \cos nx\, dx$, $\int x^m \sin nx\, dx$, $\int x^m\, e^{nx}dx$, when $m \leq 2$ using Bernoulli's formula. Simple Problems.

4.3 DEFINITE INTEGRALS

Definition of definite integral, properties of definite integrals. Simple problems.

4.1 INTEGRATION BY PARTS

Introduction:

When the integrand is a product of two functions and the method of decomposition or substitution cannot be applied, then the method of by parts is used.In differentiation , we have seen.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\text{(ie) } d(uv) = udv + vdu$$

Integrating both sides ;

$$\int d(uv) = \int udv + \int vdu$$

$$uv = \int udv + \int vdu$$

$$\text{(ie) } \int udv = uv - \int vdu$$

$\therefore \int udv = uv - \int vdu$ is called Integration By Parts formula

The above formula is used by taking proper choice of 'u' and 'dv' 'u' should be chosen based on the following order of preference

1. Inverse trigonometric functions
2. Logarithmic functions
3. Algebraic functions
4. Trigonometric function
5. Exponential Functions

Example:

1. Evaluate $\int x \cos x \, dx$

Solution :

$\int u dv = uv - \int v du$

Choosing $u = x$ and $dv = \cos x \, dx$

$du = dx$ $\qquad \int dv = \int \cos x dx$

$v = \sin x$

$\therefore \int x \cos x \, dx = x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + c$

2. Evaluate $\int x^2 \sin x \, dx$

Here we have to apply the integration by parts formula twice

Solution :

$\int u dv = uv - \int v du$

Choosing $u = x^2$ and $dv = \sin x \, dx$

$du = 2x \, dx$ $\qquad \int dv = \int \sin x dx$

$v = -\cos x$

$\therefore \int x^2 \sin x \, dx = x^2 (-\cos x) - \int (-\cos x) 2x \, dx$

$= -x^2 \cos x + 2\int x \cos x \, dx$

$= -x^2 \cos x + 2I$, Where $I = \int x \cos x \, dx$

Choosing $u = x$ and $dv = \cos x \, dx$

$du = dx \qquad \int dv = \int \cos x dx$

$v = \sin x$

$I = \int x \cos x\, dx \qquad = x\sin x - \int \sin x\, dx$

$= x\sin x - (-\cos x)$

$= x\sin x + \cos x)$

$\int x^2 \sin x\, dx = -x^2 \cos x + 2[x\sin x + \cos x] + c$

4.1 WORKED EXAMPLES

PART – A

1. Evaluate $\therefore \int xe^x dx$

Solution:

$\int u dv = uv - \int v du$

Choosing $u = x$ and $dv = e^x dx$

$du = dx \qquad \int dv = \int e^x dx$

$v = e^x$

$\therefore \int x e^x dx \qquad = x e^x - \int e^x dx$

$= x e^x - e^x + c$

2. Evaluate $\int x \sin x\, dx$

$\int u dv = uv - \int v du$

Choosing $u = x$ and $dv = \sin x\, dx$

$du = dx \qquad \int dv = \int \sin x dx$

$v = -\cos x$

$\therefore \int x \sin x\, dx \qquad = x(-\cos x) - \int(-\cos x)\, dx$

$= -x \cos x + \int \cos x\, dx$

$= -x \cos x + \sin x + c$

PART - B

1. Evaluate $\int x \log x \, dx$

Solution :

$$\int u\,dv = uv - \int v\,du$$

Choosing $u = \log x$ and $dv = x\,dx$

$$du = \frac{1}{x}\,dx \qquad \int dv = \int x\,dx$$

$$v = \frac{x^2}{2}$$

$$\therefore \int x \log x \, dx = \log x \; \frac{x^2}{2} - \int \frac{x^2}{2} \; \frac{1}{x} dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \; \frac{x^2}{2} + C$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4} + C$$

2. Evaluate. $\int x^2 e^{ax}\, dx$

Solution:

$$\int u\,dv = uv - \int v\,du$$

Choosing $u = x^2$ and $dv = e^{ax}\,dx$

$$du = 2x\,dx \qquad \int dv = \int e^{ax}\,dx$$

$$v = \frac{e^{ax}}{a}$$

$$\therefore \int x^2 e^{ax}\,dx = \frac{x^2 e^{ax}}{a} - \int \frac{e^{ax}}{a} \; 2x\,dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\int xe^{ax}\,dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2}{a}\ I \quad \text{Where } I=\int x\, e^{ax}\,dx$$

Choosing $u = x$ and $dv = e^{ax}$

$du = dx$ $\quad\int dv = \int e^{ax}\,dx$

$$v = \frac{e^{ax}}{a}$$

$$I= \int x\, e^{ax}\,dx = x\,\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\,dx$$

$$= \frac{x\,e^{ax}}{a} - \frac{1}{a}\int e^{ax}\,dx$$

$$= \frac{x\,e^{ax}}{a} - \frac{1}{a}\,\frac{e^{ax}}{a}$$

$$\therefore \int x^2\, e^{ax}\,dx = \frac{x^2e^{ax}}{a} - \frac{2}{a}\left[\frac{x\,e^{ax}}{a} - \frac{e^{ax}}{a^2}\right] + C$$

3. Evaluate $\int \log x\,dx$

Solution:

$$\int u\,dv = uv - \int v\,du$$

Choosing $u = \log x$ and $dv = dx$

$du = \frac{1}{x}\,dx$ $\quad\int dv= \int dx$

$$v = x$$

$$\therefore \int \log x\,dx = \log x\ x - \int x\,\frac{1}{x}\,dx$$

$$= x\log x - \int dx$$

$$= x\log x - x + c$$

4. Evaluate $\int(x+3)\cos 5x\,dx$

Solution:

$$\int u\,dv = uv - \int v\,du$$

Choosing $u = x+3$ and $dv = \cos 5x\,dx$

$du = dx$ $\qquad \int dv = \int \cos 5x\,dx$

$$v = \frac{\sin 5x}{5}$$

$$\therefore \int(x+3)\cos 5x\,dx = (x+3)\frac{\sin 5x}{5} - \int \frac{\sin 5x}{5}dx$$

$$= \frac{(x+3)\sin 5x}{5} - \frac{1}{5}\int \sin 5x\,dx$$

$$= \frac{(x+3)\sin 5x}{5} - \frac{1}{5}\left(\frac{-\cos 5x}{5}\right) + c$$

$$= \frac{(x+3)\sin 5x}{5} + \frac{\cos 5x}{25} + c$$

5. Evaluate $\int x^n \log x\,dx$

Solution:

Choosing $u = \log x$ and $dv = x^n\,dx$

$du = \frac{1}{x}\,dx$ $\qquad \int dv = \int x^n\,dx$

$$v = \frac{x^{n+1}}{n+1}$$

$$\int x^n \log x\,dx = \log x\,\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\frac{1}{x}dx$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{1}{n+1}\,\frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}\log x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

4.2 BERNOULLI'S FORM OF INTEGRATION BY PARTS FORMULA

If u and v are functions of x, then Bernoulli's form of integration by parts formula is

$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots\ldots..$, Where u', u'', u'''**....** are successive differentiation of the function u and v_1, v_2, v_3**....** the successive integration of the function v.

Example:

Evaluate $\int x^2 e^{ax}\, dx$

Choosing $u = x^2$ and $dv = e^{ax}\, dx$

$u' = 2x$ $\quad \int dv = \int e^{ax}\, dx$

$$v = \frac{e^{ax}}{a}$$

$u'' = 2$ $\quad v_1 = \dfrac{e^{ax}}{a^2}$

$$v_2 = \frac{e^{ax}}{a^3}$$

$\int udv = uv - u'v_1 + u''v_2 0 - u'''v_3 + \ldots\ldots..$,

$$\int x^2 e^{ax}\, dx = \frac{x^2 e^{ax}}{a} - \frac{2x\ e^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c$$

Note: The function 'u' gets differentiated till its derivative becomes a constant.

4.2 WORKED EXAMPLE

PART – B

1. Evaluate $\int x^2 \cos x dx$

Solution:

$\int udv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots\ldots..$,

Choosing $u = x^2$ and $dv = \cos x\, dx$

$u' = 2x$ $\quad \int dv = \int \cos x\, dx$

$$v = \sin x$$

$$u'' = 2 \qquad v_1 = -\cos x$$

$$v_2 = -\sin x$$

$$\therefore \int x^2 \cos x \, dx = x^2 \sin x - 2x(-\cos x) + 2(-\sin x) + C$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

2. Evaluate $\int x^3 e^{2x} dx$

Solution:

$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots\ldots..,$

Choosing $u = x^3$ and $du = e^{2x} dx$

$u' = 3x^2$ $\qquad \int dv = \int e^{2x} dx$

$u'' = 6x$ $\qquad v = \dfrac{e^{2x}}{2}$

$u''' = 6$ $\qquad v_1 = \dfrac{e^{2x}}{4}$

$$v_2 = \frac{e^{2x}}{8}$$

$$v_3 = \frac{e^{2x}}{16}$$

$$\int x^3 e^{2x} dx = \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{6xe^{2x}}{8} - \frac{6e^{2x}}{16} + c$$

4.3 DEFINITE INTEGRALS

Definition of Definite Integrals:

Let $\int f(x)dx = F(x) + C$, Where C is the arbitrary constant of integration The value of the integral

when x = b, is F (b) + C …1

and when x = a, is F (a) + C …2

Subtracting (2) from (1) we have

F(b) – F(a) = (The value of the integral when x=b)

- (The value of the integral when x=a)

(ie) $\int_a^b f(x)dx = [F(x)+c]_a^b$

$$= [F(b) + c] - [F(a) +c]$$

$$= [F(b) +c] – [F(a) +c]$$

$$= F(b) – F(a)$$

Thus $\int_a^b f(x)\,dx$ is called the definite integral, here a and b are called the lower limit and upper limit of integral respectively.

Properties of Definite integrals

Certain properties of definite integral are useful in solving problems. Some of the often used properties are given below . It is assumed throughout that f′(x) = F(x)

1 $\int_a^a f(x)dx = 0$

2 $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3 $\int_a^b k\,dx = k(b-a)$ where k is a constant

4 $\int_a^b kf(x)dx = k\int_a^b f(x)\,dx$

5 $\int_a^b [f(x) \pm g(x)]dx = \int_b^a f(x)\,dx \pm \int_a^b g(x)\,dx$

6 If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

7 If $f(x) \le 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \le 0$

8 If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

9 If $a < c < b$ in $[a,b]$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

10 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(i.e) value of the integral is independent of the variable of integration.

11 $\int_0^a f(x)dx = \int_o^a f(a-x)dx$

12 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

13 $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(x)$ is even i.e $f(-x) = f(x)$

$= 0$ if $f(x)$ is odd i.e $f(-x) = f(x)$

14 $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(2a-x)\,dx$

Examples:

1. Evaluate $\int_{25}^{9} \sqrt{x}\,dx$ if $\int_9^{25} \sqrt{x}\,dx = \frac{196}{3}$

Solution:

$$\int_{25}^{9} \sqrt{x}\,dx = -\int_9^{25} \sqrt{x}\,dx \quad \text{using property } \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$= -\frac{196}{3}$$

2. Given that $\int_2^7 f(x)\,dx = 20$ and $\int_4^7 f(x)\,dx = 13$, find $\int_2^4 f(x)\,dx$

Solution:

Using $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ if $a < c < b$,

We have $\int_2^7 f(x)dx = 20 = \int_2^4 f(x)\ dx + \int_4^7 f(x)dx$

$$20 = \int_2^4 f(x)\ dx + 13$$

$$\therefore \int_2^4 f(x)\ dx = 20 - 13 \quad = 7$$

3. Evaluate $\int_1^3 \sqrt[3]{x}\ dx$

Solution:

$$\int_1^3 \sqrt[3]{x}\ dx = \int_1^3 x^{\frac{1}{3}} dx$$

$$= \left[\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} \right]_1^3 = \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}} \right]_1^3$$

$$= \frac{3}{4}\left[x^{\frac{4}{3}} \right]_1^3 = \frac{3}{4}\left[3^{\frac{4}{3}} - 1 \right]$$

WORKED EXAMPLES

PART – A

1. Evaluate $\int_1^2 \frac{dx}{x}$

Solution:

Let $I = \int_1^2 \frac{dx}{x}$

$= [\log x]_1^2$

$= \log 2 - \log 1 \qquad (\because \log 1 = 0)$

$I = \log 2$

2. Evaluate $\int_0^{\frac{\pi}{2}} \sin x dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sin x dx$

$= [-\cos x]_0^{\pi/2}$

$= -\cos\frac{\pi}{2} + \cos 0$

$= 0+1$

$I = 1$

3. Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

Solution:

Let I = $\int_0^{\frac{\pi}{4}} \sec^2 x dx$

$= [\tan x]_0^{\frac{\pi}{4}}$

$= \tan \frac{\pi}{4} - \tan 0$

= 1-0

I = 1

4. Evaluate : $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

Let I = $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= [\sin^{-1} x]_0^1$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2} - 0$

$I = \frac{\pi}{2}$

5. Evaluate $\int_1^2 (x - x^2)dx$

Solution:

Let I = $\int_1^2 (x - x^2)dx$

$$\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$\left[\frac{2^2}{2} - \frac{2^3}{3}\right] - \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$\left[2 - \frac{8}{3}\right] - \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$-\frac{2}{3} - \frac{1}{6} = \frac{-4-1}{6}$$

$$= -\frac{5}{6}$$

6. Evaluate $\int_{-1}^{1} x^2 dx$

Solution:

$f(x) = x^2$

Now $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is an even function

$\int_{-1}^{1} x^2 dx \quad = 2\int_0^1 x^2 dx$ Using property of even function

$$= 2\left[\frac{x^3}{3}\right]_0^1 \qquad =$$

$$= 2\left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \frac{2}{3}$$

PART - B

1. Evaluate: $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} dx \qquad \left[\because \cos^2 x = \frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x) dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} - \left\{0 + \frac{\sin 2(0)}{2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin \pi}{2} - \{0+0\}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right]$$

$$I = \frac{\pi}{4}$$

2. Evaluate : $\int_0^{\frac{\pi}{2}} \text{Sin}^3 x\, dx$

Solution:

$$\text{Let I} = \int_0^{\frac{\pi}{2}} \text{Sin}^3 x\, dx \qquad \left| \begin{array}{l} \sin 3x = 3\sin x - 4\sin^3 x \\ \therefore \sin^3 x = \frac{1}{4}[3\sin x - \sin 3x] \end{array} \right.$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} [3\sin x - \sin 3x] dx$$

$$= \frac{1}{4}\left[-3\cos x - \left(\frac{-\cos 3x}{3}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[-3\cos\frac{\pi}{2} + \frac{\cos 3\left(\frac{\pi}{2}\right)}{3} - \left\{-3\cos 0 + \frac{\cos 0}{3}\right\}\right]$$

$$= \frac{1}{4}\left[0 + 0 - \left\{-3(1) + \frac{1}{3}\right\}\right]$$

$$= \frac{1}{4}\left[0 + 3 - \frac{1}{3}\right] = \frac{1}{4}\left[\frac{9-1}{3}\right]$$

$$= \frac{1}{4}\left(\frac{8}{3}\right) = \frac{2}{3}$$

3. Evaluation : $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x} dx$

Solution:

$$\text{Let} \quad = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1+\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1+\sin x)(1-\sin x)}{(1+\sin x)} dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin x) dx$$

$$= [x-(-\cos x)]_0^{\frac{\pi}{2}}$$

$$= [x+(\cos x)]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}+\cos\frac{\pi}{2}-(0+\cos 0)$$

$$= \frac{\pi}{2}+0-(0+1)$$

$$I = \frac{\pi}{2}-1$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \text{Sin}^3 x \cos x \, dx$

Solution:

$$\text{Let } I = \int \text{Sin}^3 x \cos x \, dx$$

Put u = sin x

du = cosx dx

$$= \int u^3 du$$

$$= \frac{u^4}{4} \quad = \quad \frac{\sin^4 x}{4}$$

Now $\int_0^{\frac{\pi}{2}} \text{Sin}^3 x \cos x \, dx = \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{2}}$

$$= \frac{1}{4}\left[\left(\sin\frac{\pi}{2}\right)^4 - (\sin 0)^4\right]$$

$$= \frac{1}{4}\left[(1)^4 - 0\right] \qquad = \frac{1}{4}$$

5. Evaluate : $\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x \, dx$

Solution:

Let $\int \cos 5x \sin 3x dx$ cos A sin B

$= \frac{1}{2}\int [\sin(5x + 3x) - \sin(5x - 3x)] dx$ $= \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

$$= \frac{1}{2}\int [\sin 8x - \sin 2x] \, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 8x}{8} - \frac{(-\cos 2x)}{2}\right]$$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right]$$

Now $\int_0^{\frac{\pi}{2}} \cos 5x \sin 3x \, dx$

$$= \frac{1}{2}\left[-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[-\frac{\cos 8\left(\frac{\pi}{2}\right)}{8}+\frac{\cos 2\left(\frac{\pi}{2}\right)}{2}-\left\{-\frac{\cos 0}{8}+\frac{\cos 0}{2}\right\}\right]$$

$$=\frac{1}{2}\left[-\frac{\cos 4\pi}{8}+\frac{\cos \pi}{2}-\left\{-\frac{1}{8}+\frac{1}{2}\right\}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{8}-\frac{1}{2}+\frac{1}{8}-\frac{1}{2}\right]$$

$$=\frac{1}{2}[-1] = -\frac{1}{2}$$

6. Evaluate $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx$

Solution:

Using $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$ We get

$$I = \int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx \qquad \ldots(1)$$

$$= \int_0^9 \frac{\sqrt{9-x}}{\sqrt{9-x}\,\sqrt{9-(9-x)}}\,dx$$

$$I \quad \int_0^9 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}\,dx \qquad \ldots(2)$$

Adding (1) and (2)

$$2I = \int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx + \int_0^9 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}}\,dx$$

$$= \int_0^9 \frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\,dx$$

$$= \int_0^9 1dx = [x]_0^9$$

$$= 9-0$$

$$2I = 9$$

$$I = \frac{9}{2}$$

7. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

Solution:

Using the result $\int_0^a f(a)dx = \int_0^a f(a-x)\,dx$ We get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx \quad \ldots(1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx \quad \ldots(2)$$

Adding (1) and (2)

$$2\,I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} 1dx \quad = [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0$$

$$2I \quad = \frac{\pi}{2}$$

$$I \quad = \frac{\pi}{4}$$

8. Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \, dx$

Solution:

f(x) = x cos x

f(-x) = (-x) cos (-x) = -x cos x

= -f(x)

$\therefore$ f(x) is an odd function

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x \, dx = 0$ Using property of odd function

EXERCISE

PART – A

1. Find the value of $\int xe^{2x} dx$
2. Evaluate $\int x \cos x \; dx$
3. Evaluate $\int xe^{-x} \; dx$
4. Evaluate the following

   (i) $\int_0^1 \frac{dx}{5-3x}$ (ii) $\int_0^1 (5-3x)dx$

(iii) $\int_{1}^{2}\left(x+2x^2\right)dx$

(iv) $\int_{0}^{\frac{\pi}{2}}\cos x\ \ dx$

(v) $\int_{0}^{4}\frac{dx}{\sqrt{5}-x}$

(vi) $\int_{0}^{\pi}\sin x dx$

(vii) $\int_{0}^{\pi}\cos x dx$

(viii) $\int_{0}^{\frac{\pi}{4}}\tan^2 x dx$

(ix) $\int_{0}^{1}\frac{dx}{1+x^2}$

(x) $\int_{0}^{1}\frac{e^x}{1+e^x}dx$

PART – B

1. Integrate the following with respect t x :

(i) x cos nx

(ii) $(2x-1)\ e^{2x}$

(iii) $x^2 \sin x$

(iv) $x^2\ e^{3x}$

(v) $x \cos^2 x$

(vi) $x^3 \log x$

(vii) $x^2\ e^{-x}$

(viii) $x \sec^2 x$

2. Evaluate the following

(i) $\int_{0}^{\frac{\pi}{2}}\sin^2 x dx$

(ii) $\int_{0}^{\frac{\pi}{2}}\cos^3 x dx$

(iii) $\int_{0}^{\frac{\pi}{2}}\frac{\cos^2 x}{1-\sin x}\ dx$

(iv) $\int_{0}^{\frac{\pi}{2}}\frac{\sin^2 x}{1+\cos x}\ dx$

(v) $\int_{0}^{\frac{\pi}{2}}\frac{\sin^2 x}{1-\cos x}\ dx$

(vi) $\int_{0}^{\frac{\pi}{2}}\left(2+\sin x\right)^3\cos x\,dx$

(vii) $\int_{0}^{\frac{\pi}{2}}\cos^5 x\sin x\,dx$

(viii) $\int_{0}^{\frac{\pi}{2}}\sqrt{\sin x}\cos x\,dx$

(ix) $\int_0^{\frac{\pi}{6}} \sin 2x \cos 3x \, dx$ (x) $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

(xi) $\int_0^1 (2x+3)^4 dx$ (xii) $\int_0^2 x^2 \sqrt{x^3+1} \, dx$

(xiii) $\int_0^{\frac{\pi}{4}} (\cos 2x + \sin 4x) dx$ (xiv) $\int_0^1 \frac{dx}{1+e^{-x}}$

(xv) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x} \, dx$ (xvi) $\int_0^{\frac{\pi}{2}} \sin^7 x \cos x dx$

(xvii) $\int_0^1 \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx$ (xviii) $\int_0^{\pi/2} (\sin x + \cos x)^2 dx$

(xix) $\int_0^{\frac{\pi}{2}} \sqrt{1+\sin x} dx$ (xx) $\int_0^1 (3x^2+1)(x-1) \, dx$

3. (i) Evaluate $\int_1^5 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{5-x}} \, dx$

(ii) Evaluate $\int_0^a x^2 (a-x)^{\frac{3}{2}} \, dx$

(iii) Evaluate $\int_0^1 x(1-x)^3 \, dx$

ANSWERS

PART – A

1. $\frac{xe^{2x}}{2} - \frac{1}{4} e^{2x} + c$

2. xsinx +cos x +c

3 .$-xe^{-x}-e^{-x}+c$

4 .(i) $\frac{1}{3}\log\left(\frac{5}{2}\right)$ (ii) $\frac{7}{2}$ (iii) $\frac{37}{6}$

(iv) 1 (v) $\log\frac{\sqrt{5}-1}{\sqrt{5}-4}$ (vi) 2

(vii) 0 (viii) $1-\frac{\pi}{4}$ (ix) $\frac{\pi}{4}$

(x) $\log\left(\frac{1+e}{2}\right)$

PART – B

1 (i) $\frac{x}{n}\sin n x+\frac{1}{n^2}\cos nx+c$

(ii) $e^{2x}(x-1)=c$

(iii) $-x^2\cos x+2x\sin x+2\cos x+c$

(iv) $\frac{e^{3x}}{3}\left(x^2-\frac{2x}{3}+\frac{2}{9}\right)+c$

(v) $\frac{x^2}{4}+x\sin 2x+\frac{1}{4}\cos 2x+c$

(vi) $\frac{x^4\log x}{4}-\frac{x^4}{16}+c$

(vii) $-x^2e^{-x}-2xe^{-x}\,2e^{-x}+c$

(viii) x tanx - log sec x +c

2. (i) $\frac{\pi}{4}$ (ii) $\frac{2}{3}$ (iii) $\frac{\pi}{2}+1$

(iv) $\frac{\pi}{2}-1$ (v) $\frac{\pi}{2}+1$ (vi) $\frac{65}{4}$

(vii). $\frac{1}{6}$ (viii) $\frac{2}{3}$ (ix) $\frac{3\sqrt{3}-4}{10}$

(x) $\frac{1}{2}$ (xi) 288.2 (xii) $\frac{52}{9}$

(xiii) 1 (xiv) $\log \frac{1+e}{2}$ (xv) log 2

(xvi) $\frac{1}{8}$ (xvii) $\log \frac{e^2+1}{2e}$ (xviii) $\frac{x}{2}+1$

(xix) 2 (xx) 4

3. (i) $\frac{3}{2}$ (ii) $\frac{16a^{a/2}}{3/5}$ (iii) $\frac{1}{20}$

UNIT – V

PROBABILITY DISTRIBUTION – 1

RANDOM VARIABLE

5.1. Definition of Random Variable – Type –Probability Mass Function – Probability density function. Simple problems.

5.2. Mathematical expectation of discrete variable, mean and variance. Simple problems.

BINOMIAL DISTRIBUTION

5.3. Definition

$$P(X = X) = \begin{cases} nCxp^{x}q^{n-x} & , x = 0,1,2,......., n \\ 0 & , \text{ Otherwise} \end{cases}$$

(Statement only) Expressions for mean and variance, Simple Problems

5.1 RANDOM VARIABLE

INTRODUCTION

Let a coin be tossed. Nobody knows what we will get whether a head or tail. But it is certain that either a head or tail will occur. In a similar way, if a dice is thrown, we may get any of the faces 1,2,3,4,5, and 6. But nobody knows which one will occur. Experiments of this type where the outcome cannot be predicted are called 'random' experiments.

The word probability or chance is used commonly in day –to-day life. For example the chances of India and South Africa winning the world cup cricket, before the start of the game are equal (i.e., 50:50). We often say that it will rain tomorrow. Probably I will not come to function today. All these terms – chance, probable, etc., convey the same meaning i.e., that event is not certain to take place. In other words, there is an uncertainty about the happening of the event. The term probability refers to the randomness and uncertainty.

TRAIL AND EVENT

Consider an experiment of throwing a coin. When tossing a coin, we may get a head (H) or tail (T). Here tossing of a coin is a trail and getting a head or tail is an event.

From a pack of cards, drawing any three cards is trail and getting a king or a queen or a jack are events.

Throwing of a dice is a trail and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Sample space:

The set of all possible cases of an experiment is called the sample space and is denoted by S

Mathematical Definition of Probability

$$\text{Probability of happening an event E}$$
$$= \frac{\text{Number of favourable cases of the event}}{\text{Total number of exhaustive cases}}$$
$$= \frac{m}{n}$$

Where m = number of favorable cases = n(E)

n = number of exhaustive cases = n(S)

Random Variable:

A function x which transforms events of a random experiment into real numbers is called random variable. It is denoted as X: S→R where S is sample space of random experiment and R is set of real numbers

Example:

Two coins are tossed at a time

Sample space is S = {HH, HT, TH,TT}

If we take X is the number of heads appearing then HH becomes 2, HT and TH becomes 1 and TT becomes 0

$\therefore$ X (number of heads) is a random variable

TYPES OF RANDOM VARIBLES

The are two types of random variables known as

(i) Discrete random variable

(ii) Continuous random variable

Discrete random variable

If a random variable takes only a finite or a countable number of values, it is called a discrete random variable.

For example, when two coins are tossed the number of heads obtained is the random variable X. X assumes the values 0,1,2 which is a countable set. Such a variable is called discrete random variable.

Definition: probability Mass Function:

Let X be a discrete random variable with values $x_1, x_2, x_3, \ldots\ldots. x_n$. Let p ($x_i$) be a number associated with each x_i

Then the function p is called the probability function of X if it satisfies the conditions:

(i) $p(x_i) \geq 0$ for $i = 1, 2, 3, \ldots. n$

(ii) $\sum p(x_i) = 1$

The set of ordered pairs (x_i, $p(x_i)$) is called the probability distribution of X. The probability function is also known as Probability Mass Function of X.

Continuous Random Variable:

A random variable X is said to be continuous if it can take all possible values between certain limits.

Examples: 1. Life time of electric bulb in hours

2. Height, weight, temperature, etc,.

Definition: Probability density function:

A function f is said to be probability density function (pdf) of the continuous random variable X if it satisfies the following conditions:

1. $f(x) \geq 0$ for all $x \in R$;
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Definition:

Distribution function (Cumulative Distribution Function):

The function F(x) is said to be the distribution function of the random variable X if $F(x) = P(X \leq x); -\infty \leq x \leq \infty$.

The distribution function F is also called Cumulative distribution function.

Note:

1. If X is a discrete random variable then from the definition it follows that $F(x) = \Sigma\, p(x_i)$ where the summation is overall X_I such that $x_i \leq x$.
2. If X is a continuous random variable, then from the definition it follows that

$F(x) = \int_{-\infty}^{x} f(t)dt \quad -\infty \leq x \leq \infty$. where f(t) is the value of the probability density function of X at t.

WORKED EXAMBLES
PART - A

1. Find the probability distribution of X when tossing a coin, when X is defined as setting a head.

Solution:

Let X denote getting a head.

$$\text{Probability of getting a head} = \frac{1}{2}$$

$$\text{Probability of getting a tail} = \frac{1}{2}$$

$\therefore$ The probability distribution of X is given by

x	0	1
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

2. In a class of 10 students, 4 are boys and the rest are girls. Find the probability that a student selected will be a girl.

Solution:

Total number of students = 10

Number of boys = 4

Number of girls = 6

$$\text{Probability that a girl is selected} = \frac{m}{n}$$

$$= \frac{6}{10} = \frac{3}{5}$$

3. When throwing a die what is probability of getting a 4?

Solution:

Total number of cases n = 6 $\{1,2,3,4,5,6,\}$

Number of favorable cases = 1

$$\text{Probability of getting } 4 = \frac{m}{n} = \frac{1}{6}$$

4. Find the chance that if a card is drawn at random from an ordinary pack, it is one of the court cards.

Solution:

Total no. of exhaustive cases = 52 cards.

Number of favorable cases = 12

(Court cards mean kings, queens, jacks. There are $4 \times 3 = 12$ court cards)

$$\text{Probability} = \frac{12}{52} = \frac{3}{13}$$

5. A bag contains 7 white and 9 red balls. Find the probability of drawing a white ball.

Solution:

Number of favorable cases = 7

Number of exhaustive cases = 16

$$\text{Probability} = \frac{7}{16}$$

(One white ball can be drawn out of 7 in $7c_1$ ways = 7 ways)

6. Verify that $f(x)=\begin{cases}\dfrac{2x}{9}, & 0\le x\le 3\\ 0, & \text{otherwise}\end{cases}$ is a probability density function.

Solution:

$$\int_{-\infty}^{\infty} f(x)dx=\int_0^3 \frac{2x}{9}dx=\frac{2}{9}\left(\frac{x^2}{2}\right)_0^3=\frac{2}{9}.\frac{9}{2}=1$$

$\Rightarrow$ f (x) is a probability density function

7. A continuous random variable X has the pdf defined by

$$f(x)=\begin{cases}ce^{-ax}, 0<x<\infty\\ 0, \text{ else where}\end{cases}$$

Find the value of c if $a>0$.

Solution:

$$\int_{-\infty}^{\infty} f(x)dx=c\int_0^{\infty} e^{-ax}=c\left(\frac{e^{-ax}}{-a}\right)_0^{\infty}=c\left[0-\frac{1}{-a}\right]=\frac{c}{a}$$

Since pdf is given $\int_{-\infty}^{\infty} f(x)dx=1$

$$\frac{c}{a}=1$$

$$\Rightarrow c = a$$

PART - B

1. Find the probability mass function, and the cumulative distribution function for getting '3' when two dice are thrown.

Solutions:

Let X is the random variable of setting number of '3's. Therefore X can take the values 0,1,2.

Two dice are thrown, therefore total number of exhaustive cases is 36.

S={(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)

(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

$$\text{Probability of no '3'} = P\,(X = 0) = \frac{25}{36}$$

$$\text{Probability of one '3'} = P\,(X = 1) = \frac{10}{36}$$

$$\text{Probability of two '3'} = P\,(X = 2) = \frac{1}{36}$$

Probability mass function is given by

X	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Cumulative distribution function

$$F(x) = \sum_{x=-\infty}^{x} P(X = xi)$$

$$F(0) = P(X = 0) = \frac{25}{36}$$

$$F(1) = P(X = 0) + P(X = 1) = \frac{25}{36} + \frac{10}{36} = \frac{35}{36}$$

$$F(2) \quad = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{25}{36} + \frac{10}{36} + \frac{1}{36} = \frac{36}{36} = 1$$

X	0	1	2
$F(x)$	$\frac{25}{36}$	$\frac{35}{36}$	1

2. A random variable X has the following probability mass function

x	0	1	2	3	4	5	6
p(x)	k	3k	5k	7k	9k	11k	13k

Find (i) k (ii) Evaluate P (X < 4), P (X ≥ 5), and P (3 < x ≤ 6)

(iii) What is the smallest value of x for which $P(X \le x) > \frac{1}{2}$

Solutions:

i. Since P(X=x) is the probability mass function,

$$\sum_{i=0}^{6} p_i = 1$$

i.e, k+3k+7k+9k+11k+13k=1

49k = 1

$$k = \frac{1}{49}$$

ii $P(X < 4) = P(X = 0) + P(X=1) + P(X=2) + P(X=3)$

$$= k + 3k + 5k + 7k$$

$$= 16k = \frac{16}{49}$$

$$P(X \ge 5) = P(X = 5) + P(X = 6) = 11k + 13k = 24k = \frac{24}{49}$$

$$P(3 < x \le 6) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 9k + 11k + 13k = 33k = \frac{33}{49}$$

iii The minimum value of x may be determined by trial and error method.

$$P(X \le 0) = k = \frac{1}{49} < \frac{1}{2}$$

$$P(X \le 1) = k + 3k = 4k = \frac{4}{49} < \frac{1}{2}$$

$$P(X \le 2) = k + 3k + 5k = 9k = \frac{9}{49} < \frac{1}{2}$$

$$P(X \le 3) = k + 3k + 5k + 7k = 16k = \frac{16}{49} < \frac{1}{2}$$

$$P(X \le 4) = k + 3k + 5k + 7k + 9k = 25k = \frac{25}{49} > \frac{1}{2}$$

$\therefore$ The smallest value of x for which $P(X \le x) > \frac{1}{2}$ is 4

3. A continuous random variable X has pdf

$$f(x) = \begin{cases} kx(1-x)^{10} & ,0 < x < 1 \\ 0 & ,\text{otherwise} \end{cases}$$

Find k.

Solution:

Since $f(x)$ is a pdf, we have $\int_{-\infty}^{\infty} f(x)\ dx = 1$

$$\int_0^1 kx(1-x)^{10} dx = 1$$

Put $t = 1 - x \Rightarrow x = 1 - t$

$dt = -dx$

when $x = 0, t = 1$

when $x = 1, t = 0$

$$k\int_1^0 (1-t)t^{10}(-dt) = 1$$

$$-k\left[\frac{t^{11}}{11} - \frac{t^{12}}{12}\right]_1^0 = 1$$

$$-k\left[0 - \left(\frac{1}{11} - \frac{1}{12}\right)\right] = 1$$

$$k\left(\frac{12-11}{132}\right) = 1$$

$$k = 132$$

4 For the pdf $f(x) = \begin{cases} cx\ (1-x)^3 & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$

Find (i) the constant c.

(ii) $P\left(X < \frac{1}{2}\right)$

Solution:

(i) Since f(x) is a pdf, we here $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 cx(1-x)^3 dx = 1$$

$$c\int_0^1 (1-x)(1-(1-x))^3 dx = 1 \qquad (\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx)$$

$$c\int_0^1 (1-x)x^3 dx = 1$$

$$c\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 1$$

$$c\left(\frac{1}{4} - \frac{1}{5}\right) = 1 \Rightarrow c = 20$$

$$P(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} 20x(1-x)^3\,dx$$

$$= 20\int_0^{\frac{1}{2}} x(1-3x+3x^2-x^3)dx$$

(ii) $$= 20\int_0^{\frac{1}{2}} (x-3x^2+3x^3-x^4)dx$$

$$= 20\left\{\frac{x^2}{2}-\frac{3x^2}{3}+3\frac{x^4}{4}-\frac{x^5}{5}\right\}_0^{\frac{1}{2}}$$

$$= 20\left[\frac{1}{8}-\frac{4}{8}+\frac{3}{4}\left(\frac{1}{16}\right)-\frac{1}{5}\left(\frac{1}{32}\right)\right]$$

$$= 20\left[\frac{3}{64}-\frac{1}{160}\right] = 20\left[\frac{15-2}{320}\right] = \frac{13}{16}$$

5.2 MATHEMATICAL EXPECTATION OF DISCRETE VARIABLE

Expectation of a discrete random variable.

Definition: If X denotes a discrete random variable which can assume the value $x_1,x_2,\ldots,x_n$ with respective probabilities $p_1,p_2,\ldots,p_n$ then the mathematical expectation of X, denoted by E(X) is defined by

$$E(X) = p_1x_1+p_2x_2+\ldots+p_nx_n$$

$$= \sum_{i=1}^{n} p_i x_i \quad \text{where} \quad \sum_{i=1}^{n} p_i = 1$$

Thus E(X) is the weighted arithmetic mean of the values x_i with the weight to p (x_i)

$\therefore$ mean $\overline{X} = E(X)$

Hence the mathematical expectation E(X) of a random variable is simply the arithmetic mean.

Result: If $\phi(x)$ is a function of a random variable X, then

$E[\phi(x)] = \Sigma\, P(X=x)\, \phi(x).$

Properties of mathematical expectation:

(1) $E(c)$ = c where c is a constant

(2) $E(cX)$ = $c\,E(X)$ where c is a constant.

(3) $E(aX + b)$ = $a\,E(X) + b$ where a & b are constants.

(4) Variance of X = $\text{var}(X) = E\{X - E(X)\}^2$

(5) $\text{Var}(X)$ = $E(X^2) - [E(X)]^2$

(6) $\text{Var}(X \pm c)$ = $\text{Var}(X)$ where c is a constant.

(7) $\text{Var}(aX)$ = $a^2\,\text{Var}(X)$

(8) $\text{Var}(aX + b)$ = $a^2\,\text{Var}(X)$

(9) $\text{Var}(c)$ = 0 where c is a constant.

WORKED EXAMPLE

PART - A

(1) Find the expected value of the number on a die when thrown.

Solution:

x	1	2	3	4	5	6
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(x) = \Sigma x_i\,P(x_i)$$

$$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{21}{6} = \frac{7}{2}.$$

(2) Evaluate $\text{Var}(2X \pm 3)$

We have, $\text{Var}(aX \pm b) = a^2\text{Var}(X)$

$\text{Var}(2X \pm 3) = 2^2\text{Var}(X)$

$= 4\,\text{Var}(X)$

(3) A random variable X has E (X) = 2 and E (X^2) = 8. Find its variance.

$$\begin{aligned} \text{Var }(X) &= E(X^2) - [E(X)]^2 \\ &= 8-2^2 \\ &= 8-4 \\ &= 4. \end{aligned}$$

PART - B

(4) An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws when a ball is drawn at random with replacement. Also find its mean and variance.

Solution:

Let X be the random variable of drawing number of red balls in three draws.

$\therefore$ X can take the values 0,1,2,3.

$$P\text{ (Red ball)} = \frac{3}{7};\ P\text{ (not a red ball)} = \frac{4}{7}$$

$$P(X=0) = P(WWW) = \frac{4}{7}.\frac{4}{7}.\frac{4}{7}. = \frac{64}{343}$$

$$P(X=1) = 3P(RWW)$$

$$= 3.\frac{3}{7}.\frac{4}{7}.\frac{4}{7} = \frac{144}{343}$$

$$P(X=2) = 3P(RRW)$$

$$= 3.\frac{3}{7}.\frac{3}{7}.\frac{4}{7} = \frac{3.36}{343} = \frac{108}{343}$$

$$P(X=3) = P(RRR) = \frac{3}{7}.\frac{3}{7}.\frac{3}{7} = \frac{27}{343}.$$

The required probability distribution is

x	0	1	2	3
$P(X=x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

$$\text{Mean} = E(X)$$

$$= \sum_i x_i\, p(x_i)$$

$$= 0\left(\frac{64}{343}\right)+1\left(\frac{144}{343}\right)+2\left(\frac{108}{343}\right)+3\left(\frac{27}{343}\right)$$

$$= \frac{0+144+216+81}{343}$$

$$= \frac{441}{343}$$

$$E(X^2) = \sum_i x_i^2\, p(x_i)$$

$$=0^2\left(\frac{64}{343}\right)+1^2\left(\frac{144}{343}\right)+2^2\left(\frac{108}{343}\right)+3^2\left(\frac{27}{343}\right)$$

$$= \frac{0+144+432+243}{343}$$

$$= \frac{819}{343}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{819}{343} - \left(\frac{441}{343}\right)^2$$

$$= \frac{819}{343} - \frac{194481}{117649}$$

$$= \frac{280917 - 194481}{117649}$$

$$= \frac{86436}{117649}$$

(5) A random variable X has the following distribution

X	-1	0	1	2
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Find mean and variance.

Solution:

$$\text{Mean} = E(X) = \sum_i x_i p(x_i)$$

$$= -1\left(\frac{1}{3}\right) + 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{3}\right)$$

$$= -\frac{1}{3} + 0 + \frac{1}{6} + \frac{2}{3} = \frac{02+1+4}{6} = \frac{3}{6} = \frac{1}{2}$$

$$E(X^2) = \sum_i x_i^2 p(x_i)$$

$$= (-1)^2\left(\frac{1}{3}\right) + 0^2\left(\frac{1}{6}\right) + 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{1}{6} + \frac{4}{3} = \frac{2+1+8}{6} = \frac{11}{6}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{11}{6} - \left(\frac{1}{2}\right)^2$$

$$= \frac{11}{6} - \frac{1}{4} = \frac{22-3}{12} = \frac{19}{12}$$

5.3 BINOMIAL DISTRIBUTION

Introduction:

Binomial distribution was discovered by James Bernoulli (1654-1705) in the year 1700 and was first published in 1713.

An experiment which has two mutually disjoint outcomes is called a Bernoulli trail. The two outcomes are usually called “success” and “failure”.

An experiment consisting of repeated number of Bernoulli trails is called Binomial experiment. A Binomial distribution can be used under the following conditions:

i. The number of trials is finite.
ii. The trials are independent of each other.
iii. The probability of success is constant for each trial.

Probability Function of Binomial Distribution

Let X denotes the number of success in n trial satisfying binomial distribution conditions. X is a random variable which can take the values 0,1,2,…..,n, since we may get no success, one success,…….or all n success.

The general expression for the probability of x success is given by

$P(X=x)=nC_x\, p^x q^{n-x}$,x=0,1,2,3,….,n. where p=probability of success in each trial, q=1-p

Definition: A random variable X is said to follow binomial distribution, if is probability mass function if given by

$$P(X=x)=nC_x p^x q^{n-x} : x=0,1,2,3,\ldots.,n.$$
$$0 \qquad \text{Otherwise}$$

Where n, p are called parameters of the distribution.

Constants of the binomial distribution:

Mean = np

Variance = npq

Standard Deviation = $\sqrt{npq}$

Note:

(i) $0 \le p \le 1, 0 \le q \le 1$ and $p + q = 1$

(ii) In binomial distribution mean is always greater then variance.

(iii) To denote the random variable X which follows binomial distribution with parameters n and p is X~B(n,p).

WORKED EXAMPLES

PART - A

1. Comment, if any in the following statement: The mean of a binomial distribution is 5 and its standard deviation is 3.

Solution:

given mean = 5

$np = 5 \quad \ldots 1$

Standard deviation = 3

$\sqrt{npq} = 3$

Squaring,

$npq = 9 \quad \ldots 2$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{9}{5}$$

$$q = \frac{9}{5} > 1$$

Hence, the given statement is not true.

2. Find n and p in the binomial distribution whose mean is 3 and variance is 2.

Solution:

given, mean = 3

$np = 3 \quad \ldots 1$

Variance = 2

$npq = 2 \quad \ldots 2$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{2}{3} \qquad (1)$$

$$q = \frac{2}{3}$$

$\therefore$ p = 1-q

$$= 1 - \frac{2}{3}$$

$$= \frac{1}{3}$$

Put p = $\frac{1}{3}$ in n p = 3, n x $\frac{1}{3}$ = 3 $\Rightarrow$ n = 9

3. Find the mean of the binomial distribution if $p(x) = 20c_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{20-x}$

Solution:

The binomial distribution is $P(X=x) = nc_x\, p^x q^{n-x}$

Here $n = 20$ $p = \frac{2}{5}$ $q = \frac{3}{5}$

$$\text{Mean} = np = 20 \times \frac{2}{5} = 8$$

4. Write down the binomial distribution in which $n = 8, p = \frac{3}{4}$

Solution:

Here $p = \frac{3}{4}$ $,n = 8$

q = 1 – p

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

The binomial distribution is $P(X = x) = nc_x p^x q^{n-x}$

$$= 8c_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{8-x}$$

Where x= 0,1,2,….8

5. In a binomial distribution if n=9 and $p=\frac{1}{3}$, what is the value of variance.

Solution:

given $n=9, p=\frac{1}{3}$

$$q=1-p=1-\frac{1}{3}=\frac{2}{3}$$

Variance $= npq$

$= 9.\frac{1}{3}.\frac{2}{3}$

$= 2$

6. A random variable X has the mean 6 and variance 2. If it is assumed that the distribution is binomial, find n.

Solution:

given mean = 6 variable = 2

np = 6 npq = 2

$$\frac{npq}{np}=\frac{2}{6}=\frac{1}{3} \quad \Rightarrow q=\frac{1}{3}$$

$$p=1-q \;=1-\frac{1}{3}=\frac{2}{3}$$

mean $=np$

$6 \;=n\frac{2}{3}$

$n \;=9$

PART - B

1. In tossing 10 coins, what is the chance of having exactly 5 heads.

Solution:

Let X denote number of heads

p = probability of getting a head = $\frac{1}{2}$

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

n = no of trials $= 10$

The binomial distribution is

$$P(X = x) = nc_x p^x q^{n-x}$$

P(getting exactly 5 heads) = P (X = 5)

$$= 10c_5 \left(\frac{1}{5}\right)^5 \left(\frac{1}{2}\right)^{10-5}$$

$$= \frac{63}{256}$$

2. A pair of dice is thrown 10 times, if getting a doublet is considered a success, find the probability of

(i) 4 success (ii) no success

Solution:

Let X denote getting a doublet in a throw of a dice.

(A doublet means getting a pair is (1,1),(2,2),(3,3),(4,4),(5,5),(6,6)).

$$p = P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

n = number of trials $= 10$

(i) $P(4\,\text{success}) = P(X = 4) = 10c_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6$

$$= \frac{210 \times 5^6}{6^{10}} = \frac{35}{216}\left(\frac{5}{6}\right)^6$$

(ii) $P(\text{no success}) = P(X=0) = 10c_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$

3. If the sum of mean and variance of a binomial distribution is 4.8 for 5 trials, find the distribution.

Solution:

Mean = np variance = npq

Sum of mean and variance = 4.8

np + npq = 4.8

np(1+q) = 4.8

5p(1+1-p) = 4.8 $(\therefore p + q = 1)$

P^2 - 2p + 0.96 = 0 $\Rightarrow$ p=1.2,0.8

$\therefore$ p=0.8 q=0.2 ($\therefore$p cannot be greater then 1)

The binomial distribution is $p(X = x) = 5c_x(0.8)^x(0.2)^{5-x}$

When x = 0,1,2,3,4,5.

4 If on an average 1 ship out of 10 do not arrive safely to ports. Find the mean and the standard deviation of ships returning safely out of a total of 500 ships.

Solution:

Let X denote the ships arriving safely.

$p = P(\text{safe arrival}) = \frac{9}{10}$

$q = 1 - p = 1 - \frac{9}{10} = \frac{1}{10}$

$n = 500$

$$\text{mean} = np = 500 \times \frac{9}{10} = 450$$

$$\text{variance} = npq = 500 \times \frac{9}{10} \times \frac{1}{10} = 45$$

$$S.D = \sqrt{45} = 3\sqrt{5}$$

5. In a hurdle race a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down less than 2 hurdles.

Solution:

Let X denote a player clearing the hurdle.

$$q = \text{Probability of clearing} = \frac{5}{6}$$

$$p = \text{probabiity of knocking} = 1 - \frac{5}{6} = \frac{1}{6}$$

$$n = 10$$

$$\begin{aligned} P(\text{less than 2 hurdles}) &= P(X < 2) \\ &= P(X = 0) + P(X = 1) \\ &= 10c_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} + 10c_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 \\ &= \left(\frac{5}{6}\right)^{10} + \frac{10}{6}\left(\frac{5}{6}\right)^9 \qquad = \frac{5^9}{6^{10}}(15) \end{aligned}$$

6. The overall percentage of passes in a certain examination is 80. If 6 candidates appear in the examination, what is the probability that atleast 5 will pass the examination.

Solution:

Let X denote the overall percentage of passes.

$$p = \frac{80}{100} = \frac{4}{5} \qquad q = 1 - p = 1 - \frac{4}{5} = \frac{1}{5} \qquad n = 6$$

$$P(\text{atleast } 5) = P(X \geq 5) = P(X = 5) + P(X = 6)$$

$$= 6c_5\, p^5 q^1 + 6c_6\, p^6 q^0$$

$$= 6\left(\frac{4}{3}\right)^5 \left(\frac{1}{5}\right) + \left(\frac{4}{5}\right)^6 \left(\frac{1}{5}\right)^0$$

$$= \left(\frac{4}{5}\right)^5 \left(\frac{6}{5} + \frac{4}{5}\right) = 2\left(\frac{4}{5}\right)^5.$$

7. With usual notation find 'p' for binomial random variable if n = 6 and if 9P(X=4) = P(X=2).

Solution:

The binomial distribution is $P(X=x) = nc_x p^x q^{n-x}$

Given 9 P(X = 4) = P(X = 2) and n = 6

$9\ 6c_4 p^4 q^2 = 6c_2 p^2 q^4$

$9.15\ p^4 q^2 = 15 p^2 q^4$

$9\ p^2 = q^2$

Taking square root

$3p = q$

$3p = 1-p \quad (\therefore p + q = 1)$

$4p = 1$

$p = \dfrac{1}{4}$

EXERCISE

PART - A

1. Define discrete random variable?
2. When a random variable is called a continuous random variable?
3. A random variable X has the following distribution

x	0	1	2	3	4

P(X=x)	3a	4a	6a	7a	8a

find the value of a.

4. A discrete random variable takes values 0,1,2. Also if $P(X=0)=\frac{144}{169}, P(X=1)=\frac{1}{169}$ then find the value of if P(X=2)

5. If 3 coins are tossed simultaneously and X is the number of heads. What is the value of P(X=3).

6. Four coins are tossed at a time. If X denotes the number of tails, what are the possible values of X.

7. A random variable X has the following distribution function.

X	0	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Find $P(X \leq 2)$

8. A random X has the following probability distribution function.

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the distribution function F(x).

9. Examine whether f(x) = $\frac{2x}{9}, 0 \leq x \leq 3$ can be a pdf of a continuous random variable X?

10. The probability density function of a random value X is $f(x) = Ax^2$, $0 \leq x \leq 1$, determine A.

11. If $f(x) = kx^2$, $0 \leq x \leq 3$,

0 else where

is a pdf, find the value of k?

12. Verify whether f(x) = $\frac{2}{\pi}.\frac{1}{4+x^2}, -\infty < x < \infty$ is a pdf ?

13. If E(X) = 12 and $E(X^2)$=200 what is var (X).

14. If E(X) = 3 and $E(X^2)$ = 30, what is the variance of X

15. If var (X) = 2 , what is var(5 X +7)?

16. If E(X) = 8 what is the value of E (3X).

17. A binomial distribution has mean 4 and variance $\frac{8}{3}$ find p and n.

18. A discrete random variable X has the mean 6 and variance 2. If it is assumed that the distribution is binomial find n.

19. Find the S.D of the binomial distribution $10c_x\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{10-x}$

20. "For a binomial distribution mean is 9 and variance is14". Is it possible?

21. Find the mean of the binomial distribution $16c_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{16-x}$

PART - B

1. A random variable X has the following probability distribution function.

X	0	1	2	3	4	5
$P(X = x)$	a	3a	5a	7a	9a	11a

Find (i) The value of a

(ii) P(X<4)

(iii) $P(X \geq 3)$

(iv) P(2<X<5)

2. A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7	8
$P(X = x)$	k	3k	5k	7k	9k	11k	13k	15k	17k

i. Determine the value of 'k'

ii. $P(X < 5)$

iii. $P(X \geq 4)$

iv. $P(0 \leq X < 5)$

3. The probability denity function of a random variable X is $f(x) = \begin{cases} 3x^2, 0 \leq x \leq 1 \\ 0 \quad ,\text{elsewhere} \end{cases}$

If (i) $P(X \leq a) = P(X > a)$ and (ii) $P(X > b) = 0.05$

Find the value of a & b

4. If $f(x) = \begin{cases} \frac{A}{x}, 1 < x < e^3 \\ 0, \text{otherwise} \end{cases}$ is the pdf of a random variable X, find p

$P(X > e)$.

5. The amount of bread (in hundred of ks) x, that a certain bakery is able to sell in a month is found to be a numerical valued random phenomenon specified by probability density function f(x) given by

$$f(x) = \begin{cases} Ax & \text{for} & 0 \leq x \leq 5 \\ A(10 - x) & \text{for} & 5 \leq x \leq 10 \\ 0 & & \text{elsewhere} \end{cases}$$

i. Find the value of A

ii. What is the probability that the number of kg of bread that will be sold next month is,

(a) more than 500 kg

(b) Between 250 and 750 kg?

6. Two cards are drawn from a well shuffled pack of 52 cards with replacement, find the mean and variance of the number of aces

7. The probability that a student will graduate is 0.4. Find the probability that out of 5 student (i) none (ii) one (iii) at least one will be a graduate.

8. A player tosses 3 fair coins. He wins Rs.5 if 3 heads appear, Rs.3, if 2 head appears Rs.1 if 1 head occurs. On the other head he losses Rs.15/- if all tails occurs. Find the expected gain.

9. If a random variable X has the following probability distribution

X	-1	0	1	2
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

Find (i) E(X) (ii) var (X) (iii) $E(2X+3)^2$

10. A game is played with a single fair die. A player wins Rs.20 if a 2 turns up, Rs 40 of a 4 turns up, loses Rs.30 if a 6 turns up. While he neither wins nor loses if any other face turns up. Find the expected sum of money he can win.

11. Ten coins are tossed simultaneously. Find the probability of getting exactly 2 heads.

12. In a binomial distribution having 6 independent trials the probabilities of 0 and 1 successes are 0.4 and 0.2 respectively. Find p and P(X=0)

13. With usual notation find 'p' for the binomial distribution X if n=6 and if 9P(X =4)=P(X =2)

14. Find the probability that in a family of 4 children there will be ateast 1 boy and 1 girl.

15. Ten coins are tossed simultaneously. Find the probability of getting (i) atleast seven heads (ii) exactly 7 heads and (iii) almost seven heads

16. In a binomial distribution with 5 independent trial the probability of getting 1 and 2 success are 0.6 and 0.2 respectively. Find p

17. In a large consignment of iron boxes 10% are defective . A random sample of 20 is taken for inspection. Find the probability that at the most there are 3 defective iron boxes.

18. The mean and variance of a binomial variate X with parameters n and p are 16 and 8 respectively. Find P(X =0) and P(X =1)

19. Four coins are tossed simultaneously probability distribution. Find the probability of getting at least 2 heads.

20. A random various X has the following probability distribution

X	0	1	2	3	4
$P(X = x)$	3a	4a	6a	7a	8a

Find the value of a the find $E(X^2 + X)$

ANSWERS

PART-A

3. $a = \frac{1}{28}$ 4. $\frac{24}{169}$ 5. $\frac{1}{8}$

6. 0,1,2,3,4   7. $\frac{5}{6}$

8.

x	0	1	2	3
F(x)	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

10 A = 3   11. $k = \frac{1}{9}$ 13. 56

14. 21   15. 50   16. 24

17 $p = \frac{1}{3}$ $n = 12$ 18. 9   19. $\sqrt{\frac{12}{5}}$

20. not possible   21. 8

PART - B

1. (i) $a = \frac{1}{36}$ (ii) $\frac{4}{9}$ (iii) $\frac{3}{4}$ (iv) $\frac{4}{9}$

2. (i) $a = \frac{1}{81}$ (ii) $\frac{25}{81}$ (iii) $\frac{65}{81}$ (iv) $\frac{25}{81}$

3. $a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$ $b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$ 4. (i) $A = \frac{1}{3}$ (ii) $\frac{2}{3}$

5. (i) $A = \frac{1}{25}$ (ii) (a) $\frac{1}{2}$ (b) 0.75

6. $\frac{2}{13}, \frac{24}{169}$

7. (i) 0.0776 (ii) 0.2592 (iii) 0.9224

8. Rs.0.25 9. (i) $\frac{1}{2}$ (ii) $\frac{14}{12}$ (iii) $\frac{67}{3}$

10. Rs.5 11. $\frac{45}{2^{10}}$

12. (i) $\frac{1}{13}$ (ii) $\left(\frac{12}{13}\right)^6$

13. $\frac{1}{4}$ 14. $7/8$ 15. (i) $\frac{11}{64}$ (ii) $\frac{15}{128}$ (iii) $\frac{121}{125}$

16. $p = 1/7$ 17. 0.8666

18. $n = 32$ $p = \frac{1}{2}$ (ii) $\frac{1}{2^{32}}$ (iii) $\frac{1}{2^{27}}$

19. $11/16$ 20. (i) $a = \frac{1}{28}$ (ii) $\frac{72}{7}$

MATHEMATICS – III

MODEL QUESTION PAPER - 1

Time : 3 Hrs (Maximum Marks: 75)

PART – A

(Marks: 15 x 1 = 15)

Answer any fifteen (15) questions:

1. If position vectors of the points A and B are $2\vec{i}+\vec{j}-\vec{k}$ and $5\vec{i}+4\vec{j}+3\vec{k}$ find $\left|\overrightarrow{AB}\right|$
2. If the vectors $\vec{a} = 2\vec{I} - 3\vec{J}$ and $b = -6\vec{I} + m\vec{j}$ are collinear, find the value of m.
3. Define scalar product of two vectors.
4. Find the projection of the vector $2\vec{i}+3\vec{j}-\vec{k}$ on $-2\vec{i}+4\vec{j}-\vec{k}$
5. If $\vec{a} = 2\vec{i}-\vec{j}+k$ and $\vec{b}=\vec{i}+2\vec{j}+3k$ find $\vec{a}\,x\,\vec{b}$
6. Prove that $(\vec{a}-\vec{b})\ x\ (\vec{a}+\vec{b}) = 2(\vec{a}+\overline{b})$
7. Find the value of $\left[\vec{i}, \vec{j}, \vec{k}\right]$
8. Find $\vec{i}\ x \left(\vec{j}\ x\ \vec{k}\right)$ and $(i\,x\,j)\,x\,k$
9. Evaluate $\int(3x^2 - 5\sec^2 x + 7/x)dx$
10. Evaluate $\int \sin^2 x\, dx$
11. Evaluate $\int \frac{e^x}{e^x+1}dx$
12. Evaluate $\int \frac{1}{4x^2-25}dx$
13. Evaluate $\int x\, e^x dx$

14. Evaluate $\int \log x \, dx$

15. Evaluate $\int_1^3 3x^2 + 1dx$

16. Evaluate $\int_{-2}^2 x^3 \, dx$

17. Define discrete random variable.

18. A random variable X has the following probability distribution

X :	0	1	2	3	4
P(x) :	a	5a	3a	7a	4a

Find the value of a

19. A random variable X has the following probability distribution

X :	0	1	2	3
P(x) :	1/7	2/7	1/7	3/7

Find E(X)

20. Find the mean and variance of the binomial distribution given by $P(X=x) = 10C_x \, (1/4)^x \, (3/4)^{10-x}$ when x=0,1,2……10

PART - B

(Marks : 5 x 12 = 60)

[N.B :- (1) Answer all questions choosing any two divisions from each question.

(2) All questions carry equal marks.]

21 (a) Show that the points whose position vectors $2\vec{i} + 3\vec{j} - 5\vec{k}, 3\vec{i} + \vec{j} - 2\vec{k}$ and $6\vec{i} - 5\vec{j} + 7\vec{k}$ are collinear

(b) Prove that the vectors are $\vec{a} = \vec{i} + 2\vec{j} + 5\vec{k}, \vec{b} = \vec{i} + \vec{j} - 3\vec{k}$ and $\vec{c} = 7\vec{i} - 4\vec{j} + \vec{k}$ are mutually perpendicular.

(c) A particle acted on by the forces $3\vec{i} - 2\vec{j} + 2\vec{k}$ and $2\vec{i} + \vec{j} - 3\vec{k}$ is displaced from the point $2\vec{i} + 3\vec{j} - \vec{k}$ to the point $4\vec{i} - \vec{j} + 2\vec{k}$ Find the work done.

22 (a) Find the area of the triangle formed by the points whose position vectors are

$$2\vec{i}+3\vec{j}+4\vec{k},\ 3\vec{i},+4\vec{j}+2\vec{k},4\vec{i},+2\vec{j},+3\vec{k},$$

(b) Find the magnitude of the moment about the point (1,-2,3) of a force 2i + 3j + 6k whose line of action passes through the origin

(c) If $\vec{a}=\vec{i}+\vec{j};\ \vec{b}=\vec{j}+\vec{k};\ \vec{c}=\vec{k}+\vec{i};\ \vec{d}=\vec{i}+\vec{j}+\vec{k}$ verify that $\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right)=\left[\vec{a}\ \vec{d}\ \vec{b}\right]\vec{c}-\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{d}$

23 (a) Integrate (i) $\frac{\sin x}{1+\cos x}$ (ii) Sin7x Cos5x

(b) Evaluate (i) $\int\frac{6x+5}{\sqrt{3x^2+5x+6}}dx$ (ii) $\int\frac{e^{\tan x}}{\cos^2 x}dx$

(c) Evaluate $\int\frac{1}{3x^2-13x-10}dx$

24 (a) Evaluate (i) $\int x^2\log x\,dx$ (ii) $\int x\cos 5x$

(b) Using Bernoulli's formula evaluate

(i) $\int x^2 e^{2x}dx$ (ii) $\int x^2\cos 2x\,dx$

(c) Evaluate (i) $\int_1^2 x^2-3\sqrt{x}+\frac{1}{x^2}dx$ (ii) $\int_0^{\frac{\pi}{6}}\cos^2\frac{x}{2}dx$

25 (a) A Random variable X has the following probability distribution

X	0	1	2	3	4	5
P(X)	a	3a	5a	7a	9a	11a

Find (i) Value of a (ii) P(X>3) (iii) P(1≤x≤4)

(b) The random variable X has the following probability distribution

X	0	1	2	3	4	5
P(x)	1/16	¼	3/8	3/16	1/16	1/16

Find the mean and variance

(c) A perfect cube is thrown 8 times. The occurrence of 2 or 4 is called a success, find the probability of (i) 2 success (ii) atleast 2 successes.

MATHEMATICS – III

MODEL QUESTION PAPER - 2

Time : 3 Hrs (Maximum Marks: 75)

PART – A

(Marks: 15 x 1 = 15)

1. If $\vec{a} = 3\vec{i} - \vec{j} - 4\vec{k}$, $\vec{b} = -2\vec{i} + 4\vec{j} - 3\vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, find $2\vec{a} - \vec{b} - 3\vec{c}$
2. Find the direction cosines of the vector $2\vec{i} + 3\vec{j} - 4\vec{k}$
3. If $\vec{a} = 5\vec{i} - \vec{j} - 6\vec{k}$, $b = -7\vec{i} + 3\vec{j} - 2\vec{k}$ find dot product of $\vec{a}$ and $\vec{b}$
4. State the formula to find work done by the force $\bar{f}$ in displacing the particle from the point A to B.
5. Define vector product of two vectors.
6. If $\vec{a}$ and $\vec{b}$ are the two adjacent sides of a parallelogram, find its area.
7. Define scalar product of three vectors
8. Express $(\vec{a} \times b)(c \times b)$ in the form of determinant.
9. Evaluate $\int \sec^2(3+4x)dx$
10. Evaluate $\int \sin 5x \cos 2x \, dx$
11. Evaluate $\int \frac{2x}{1+x^2} dx$
12. Evaluate $\int \frac{1}{16+x^2} dx$

13. Evaluate $\int \log x\,dx$

14. Evaluate ∫ x sjnx dx

15. Evaluate $\int_{2}^{3} 3x^2 + 4\,dx$

16. Evaluate $\int_{-2}^{2} (2x^3 + 5x)\,dx$

17. Define Random variable

18. A random variable X has the following the probability distribution

X :	1	2	3	4	5
P(X) :	1/16	5/16	3/16	3/16	¼

Find P (X<3)

19. If E(X) = 5 and E (X^2) = 35 find variance of X

20. In a binomial distribution, the mean and standard deviation are 12 and 2 respectively. Find p.

PART – B

(Marks: 5 x 12 = 60)

[N.B :- (1) Answer all questions choosing any two divisions from each question.

(2) All questions carry equal marks.]

21 (a) Show that the points given by the vectors

$4\vec{i} + 5\vec{j} + \vec{k}, -\vec{j} - \vec{k},\ 3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$ are coplanar.

(b) Find the angle between the vectors

$3\vec{i} + 4\vec{j} + 12\vec{k}$ on $\vec{i} + 2\vec{j} + 2\vec{k}$.

(c) The work done by force $\vec{F} = a\vec{i} + \vec{j} + \vec{k}$ in moving the point of application from $\vec{i} + \vec{j} + \vec{k}$ to $2\vec{i} + 2\vec{j} + 2\vec{k}$. along a straight line is given to be 5 units. Find the value of a.

22 (a) Find the angle and the unit vector perpendicular to both the vectors $\vec{a} = \hat{i}+2\hat{j}+2\hat{k}$ and $b=\hat{i}-\hat{j}-\hat{k}$.

(b) Find the moment about the point $\hat{i}+2\hat{j}-\hat{k}$ of a force represented by $3\vec{i}+\vec{k}$ acting through the point $2\hat{i}-\hat{j}-2\hat{k}$

(c) Prove that $\left[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2$

23 (a) Evaluate (i) $\int(\tan x+\cot x)^2dx$ (ii) $\int\sqrt{1+\sin 2x}\,dx$

(b) Evaluate (i) $\int\tan^4 x\sec^2 x$ (ii) $\int\frac{\sin\sqrt{x}}{\sqrt{x}}dx$

(c) Evaluate $\int\frac{4x-3}{x^2+6x+8}\,dx$

24 (a) Evaluate (i) $\int x\log x\,dx$ (ii) $\int x\cos 5x$

(b) Using Bernoulli's formula evaluate

(i) $\int x^2 e^{2x}\,dx$ (ii) $\int x^2\cos 2x\,dx$

(c) Evaluate (i) $\int_0^1\frac{e^{\tan^{-1}x}}{1+x^2}dx$ (ii) $\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin x+\cos x}dx$

25 (a) Show that f(x) = $\frac{2}{\pi}\frac{1}{1+x^2}$ $-1<x<1$, is a probability density function.

(b) A random variable X has the following probability distribution

X	:	0	1	2	3
P(X)	:	1/8	3/8	3/8	1/8

Find $E(2X+3)^2$

(c) Four coins are tossed simultaneously. What is the probability of getting (a) exactly 2 heads (b) at least two heads (c) at most two heads.

MATHEMATICS – IV

UNIT – I

COMPLEX NUMBERS – I

1.1 Definition–Conjugates–Algebra of complex numbers (geometrical proof not needed)–Real and Imaginary parts. Simple problems.

1.2 Polar form of complex number – Modulus and amplitude form multiplication and division of complex numbers in polar form. Simple Problems.

1.3 Argand plane–collinear points, four points forming square, rectangle, rhombus. Simple Problems.

Introduction

The concept of imaginary numbers has its historical origin in the fact that the solution of the quadratic equation $ax^2 + bx + c = 0$ leads to an expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which is not found meaningful when $b^2 - 4ac < 0$. This is because of the fact that the square of a real number is never negative. So it created the need of the extension of the system of real numbers. Euler was the first mathematician who introduced the symbol i for $\sqrt{-1}$ with the properties $i^2 = -1$ and accordingly a root of the equation $x^2 + 1 = 0$, also symbol of the form a + ib where a and b are real numbers is called a complex number.

Definition of Complex Numbers:

A number of the form a + ib, where a,b are real numbers and $i^2 = -1$ is called a complex number.

If $z = a + ib$, then 'a' is called the real part of z and 'b' is called the imaginary part of z and are denoted by Re(z) and Im(z) respectively.

Example:

$$z = 2 + 5i$$

$$\text{Re}(z) = 2, \ \text{Im}(z) = 5$$

Note: In the complex number a + ib

i. If a = 0, then the number is purely imaginary.
ii. If b = 0, then the number is purely real.
iii. The complex number a + ib is also denoted as the ordered pair (a,b).

Conjugate of a complex numbers:

A pair of complex numbers of the form a + ib and a – ib are called conjugate complex numbers of each other.

If z is any complex number then its conjugate is usually denoted by $\overline{z}$.

$$z\overline{z} = (a+ib)(a-ib)$$

$$z\overline{z} = a^2 + b^2$$

Algebra of complex numbers:

Addition of two complex numbers:

Let z_1=a+ib and z_2=c+id be any two complex numbers. Then the sum z_1+z_2 is defined as follows.

z_1+z_2 =a+ib+c+id

= (a+c)+ i(b+d) which is again a complex numbers.

For example, 2+3i+ (-4+7i)= (2–4)+I(3+7)=-2+10i

Difference of two complex numbers:

When any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows.

$z_1 - z_2 = z_1 + (-z_2)$

For example 6+3i-(2-i) =(6+3i)+(-2+i)

=(6-2)+i(3+1)

=4+4i

and 2-i-(6+3i)=2-i+(-6-3i)=-4-4i

Multiplication of two complex numbers:

Let z_1=a+ib and z_2= c+id then

$$z_1 z_2 = (a+ib)(c+id)$$
$$= ac+ibc+iad+i^2bd$$
$$= ac+i(ad+bc)-bd$$
$$= ac-bd+i(ad+bc)$$

For example $(3+5i)(2+6i)=(3\times2-5\times6)+i(3\times6+5\times2)$

$$= -24+i(28)$$
$$= -24+28i$$

Division of two complex numbers:

Consider any two complex numbers z_1 and z_2 where $z_2 \neq 0$, the quotient

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$
$$= \frac{ac-iad+ibc-i^2bd}{c^2-i^2d^2}$$
$$= \frac{ac+i(bc-ad)+bd}{c^2+d^2} \qquad (i^2=-1)$$
$$= \frac{ac+bd+i(bc-ad)}{c^2+d^2}$$
$$\frac{z_1}{z_2} = \frac{ac+bd}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

For example, Let $z_1 = 6+3i$ and $z_2 = 2-i$

Then $\frac{z_1}{z_2} = \frac{6+3i}{2-i} = \frac{6+3i}{2-i} \times \frac{2+i}{2+i}$

$$= \frac{12+6i+6i+3i^2}{(2)^2+(-1)^2}$$
$$= \frac{12+12i-3}{4+1}$$
$$= \frac{9+12i}{5} = \frac{9}{5} + \frac{12i}{5}$$

1.1 WORKED EXAMPLES

PART - A

1. Write the real and imaginary parts of the complex number $4 - i\sqrt{7}$

Solution:

Let $z = 4 - i\sqrt{7}, \text{Re}(z) = 4, \text{Im}(z) = -\sqrt{7}$

2. What is the complex conjugate of $4 + i\sqrt{3}$

Solution:

$$\text{Let } z = 4 + i\sqrt{3}$$
$$\bar{z} = 4 - i\sqrt{3}$$

3. What is the complex conjugate of 6i-4

Solution:

$$\text{Let } z = 6i - 4 = -4 + 6i$$
$$\bar{z} = -4 - 6i$$

4. Write the following complex numbers in a+ib form

(1) $\sqrt{-27}$ (2) $3 - \sqrt{-5}$

Solution:

(i) $\sqrt{-27} = \sqrt{(-1) \times 27} = \sqrt{27i^2} = i\sqrt{27} = 0 + i\sqrt{27}$

(ii) $3 - \sqrt{-5} = 3 - \sqrt{-1 \times 5} = 3 - \sqrt{5i^2} = 3 - i\sqrt{5}$

5. Express $(-5i)\left(\frac{1}{8}i\right)$ in the form a+bi.

Solution:

$$(-5i)\left(\frac{1}{8}i\right) = -\frac{5}{8}i^2 = -\frac{5}{8}(-1) = \frac{5}{8} + i(0)$$

6. Express the following in the standard form a+ib.

(i) 3-4i+(-7-i)

(ii) 8-6i-(-4i-7)

Solution:

(i) 3-4i+(-7-i) = (3-7)+i(-4-1)=-4+i(-5)=-4-5i

(ii) 8-6i-(-4i-7)=8-6i+ 4i+7=15-2i=15+i(-2)=15-2i

7. If $z_1 = (-1,2)$, $z_2 = (-3,4)$ find $3z_1 - 4z_2$

Solution:

$$\begin{aligned} 3z_1 - 4z_2 &= 3(-1,2) - 4(-3,4) \\ &= (-3,6) - (-12,16) \\ &= (9,-10) \end{aligned}$$

8. Find the real and imaginary parts of $\dfrac{1}{3+2i}$

Solution:

Let $z = \dfrac{1}{3+2i} \times \dfrac{3-2i}{3-2i}$

$$= \frac{3-2i}{3^2+2^2} = \frac{3-2i}{9+4}$$

$$= \frac{3}{13} + i\left(\frac{-2}{13}\right)$$

$$\operatorname{Re}(z) = \frac{3}{13}; \operatorname{Im}(z) = \frac{-2}{13}$$

PART - B

1. Find the real and imaginary parts of $\dfrac{4+5i}{3-2i}$

Solution:

$$\text{Let } z = \frac{4+5i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{12+8i+15i+10i^2}{9-4i^2} = \frac{12+23i-10}{9+4}$$

$$= \frac{2+23i}{13} = \frac{2}{13} + i\frac{23}{13}$$

$$\text{Re}(z) = \frac{2}{13}, \ \text{Im}(z) = \frac{23}{13}$$

2. Find the real and imaginary parts of $\dfrac{3}{4+3i} + \dfrac{i}{3-4i}$

Solution:

$$\text{Let } z = \frac{3}{4+3i} + \frac{i}{3-4i}$$

$$= \frac{3(4-3i)}{(4+3i)(4-3i)} + \frac{i(3+4i)}{(3-4i)(3+4i)}$$

$$= \frac{12-9i}{16-9i^2} + \frac{3i+4i^2}{9-16i^2}$$

$$= \frac{12-9i}{16+9} + \frac{3i-4}{9+16}$$

$$= \frac{12-9i}{25} + \frac{3i-4}{25}$$

$$= \frac{12-9i+3i-4}{25}$$

$$= \frac{8-6i}{25} = \frac{8}{25} + \left(\frac{-6}{25}\right)i$$

$$\text{Re}(z) = \frac{8}{25}; \ \text{Im}(z) = \frac{-6}{25}$$

3. Express $\frac{(1+i)(1+2i)}{1+3i}$ in a+ib form. Find also its conjugate.

Solution:

$$\text{Let } z = \frac{(1+i)(1+2i)}{1+3i} = \frac{1+2i+i+2i^2}{1+3i}$$

$$= \frac{1+3i-2}{1+3i} = \frac{-1+3i}{1+3i} \times \frac{1-3i}{1-3i}$$

$$= \frac{-1+3i+3i-9i^2}{1-9i^2} = \frac{-1+6i+9}{1+9} = \frac{8+6i}{10}$$

$$z = \frac{8+6i}{10} = \frac{8}{10} + \frac{6}{10}i$$

$$= \frac{4}{5} + \frac{3}{5}i$$

$$\text{conjugate of } \frac{4}{5} + \frac{3}{5}i = \frac{4}{5} - \frac{3}{5}i$$

4. Find the real and imaginary part of $\left(\frac{1-i}{1+i}\right)^3$

Solution:

$$\text{Let } z = \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{1+1}$$

$$= \frac{1-2i-1}{2} = \frac{-2i}{2} = -i$$

$$z^3 = \left(\frac{1-i}{1+i}\right)^3 = (-i)^3 = -i^3 = -i(i^2)$$

$$= -i(-1)$$

$$= i = 0 + i(1)$$

Real part $= 0$, Imaginary part $= 1$

5. Express $\frac{(1+2i)^3}{(1+i)(2-i)}$ in a+ib form.

Solution:

$$\text{Let } z = \frac{(1+2i)^3}{(1+i)(2-i)} \qquad (a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$$

$$= \frac{1+8i^3+6i+12i^2}{2-i+2i-i^2} = \frac{1-8i+6i-12}{2+i+1} = \frac{-11-2i}{3+i} \qquad i^3 = -i$$

$$z = \frac{-11-2i}{3+i} \times \frac{3-i}{3-i}$$

$$= \frac{-33+11i-6i+2i^2}{9-i^2} = \frac{-33+5i-2}{9+1}$$

$$= \frac{-35+5i}{10} = \frac{-35}{10} + \frac{5i}{10}$$

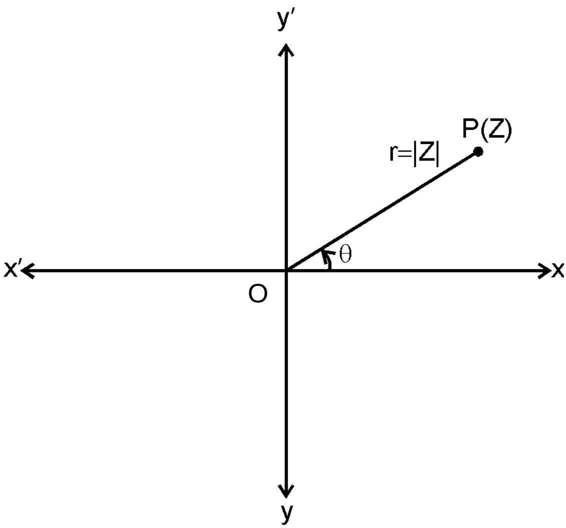
$$= \frac{-7}{2} + \frac{1}{2}i$$

$$\text{Re}(z) = \frac{-7}{2}, \quad \text{Im}(z) = \frac{1}{2}$$

1.2. POLAR FORM OF COMPLEX NUMBERS (OR) (MODULUS-AMPLITUDE FORM)

Let the point P represent the non-zero complex number $Z = a + ib$ Let the directed line segment OP be of length r and θ be the angle which OP makes with the positive direction of x-axis.

We may note that the point P is uniquely determined by the ordered pair of real numbers (r,θ), called the polar coordinates of the point P. We consider the origin as the pole and the positive direction of the x-aixs as the initial line.



We have, $a=r\cos\theta$, $b=r\sin\theta$ and therefore, $Z=a+ib=r(\cos\theta+i\sin\theta) =re^{i\theta}$. The latter is said to be the polar form of the complex number. Equating real and imaginary parts,

$a=r\cos\theta$, $b=r\sin\theta$

Squaring and adding,

$$a^2+b^2=r^2(\cos^2\theta+\sin^2\theta)$$

$$a^2+b^2=r^2$$

$$\Rightarrow r=\sqrt{a^2+b^2} \qquad \text{(Taking positive sign)}$$

$$\text{Now} \quad \cos\theta = \frac{a}{r}; \qquad \sin\theta = \frac{b}{r}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{r}}{\frac{a}{r}} = \frac{b}{a}$$

$$\Rightarrow \tan\theta = \frac{b}{a} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

r is called the modulus of the complex number Z and is denoted by $|Z|$ and θ is called the argument (or) amplitude of the complex number Z and is denoted by arg(Z) (or) amp Z. r is single valued where as 'θ' will have infinite number of values differing by multiples of 2π. The values of θ lying in the range $-\pi<\theta\le\pi$ is called the principal value of the argument.

Theorems of complex numbers:

(i) The product of two complex numbers is a complex number whose modulus is the product of their moduli and whose amplitude is the sum of their amplitudes.
i.e. $|Z_1Z_2| = |Z_1||Z_2|$ and $\arg(Z_1Z_2) = \arg Z_1 + \arg Z_2$

(ii) The quotient of two complex numbers is a complex number whose modulus is the quotient of their modulii and whose amplitude is the subtraction of the amplitude of the denominator from the amplitude of the numerator.

i.e. $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$ and $\arg\left(\frac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2)$

1.2 WORKED EXAMPLES
PART - A

1. Find the modulus of the complex number $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Solution:

Let $Z = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

Here $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$

$$\text{modulus} = r = \sqrt{a^2+b^2} = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1$$

2. Find the amplitude of the complex number $1+i\sqrt{3}$

Solution:

Let $Z_1 = 1+i\sqrt{3} = r[\cos\theta + i\sin\theta]$

$r\cos\theta = 1 = a \qquad r\sin\theta = \sqrt{3} = b$

$$\text{Amplitude} = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \qquad \theta = \frac{\pi}{3}$$

3. If $Z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$ and $Z_2 = \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}$ what is the value of Z_1Z_2.

Solution:

$$\begin{aligned} Z_1Z_2 &= \left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right) \\ &= \cos\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) + i\sin\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) \\ &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \\ &= 0 + i(1) = i \end{aligned}$$

4. If $Z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ and $Z_2 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ what is the value of $\frac{Z_1}{Z_2}$

Solution:

$$\begin{aligned} \frac{Z_1}{Z_2} &= \frac{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}} \\ &= \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}i \end{aligned}$$

PART - B

1. Find the modulus and amplitude of $\frac{1}{2}+i\frac{\sqrt{3}}{2}$

Solution: Let $Z=\frac{1}{2}+i\frac{\sqrt{3}}{2}=a+ib$

Here $a=\frac{1}{2},\ b=\frac{\sqrt{3}}{2}$

$$\text{modulus}=r=\sqrt{a^2+b^2}\quad=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\sqrt{\frac{1}{4}+\frac{3}{4}}$$

$$=\sqrt{1}\qquad=1$$

$$\text{and}\quad \theta\ =\tan^{-1}\left(\frac{b}{a}\right)$$

$$=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\tan^{-1}\left(\sqrt{3}\right)$$

$$\theta=60^\circ=\frac{\pi}{3}$$

2. Find the modulus and amplitude of $\sqrt{3}-i$

Solution:

Let $Z=\sqrt{3}-i$, Here $a=\sqrt{3},\ b=-1$

$$\text{modulus}=r=\sqrt{a^2+b^2}=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2$$

$$\cos\theta=\frac{a}{r},\quad \sin\theta=\frac{b}{r}$$

$$\cos\theta=\frac{\sqrt{3}}{2}\quad \sin\theta=-\frac{1}{2}$$

$$\therefore \theta=-30^\circ\qquad=-\frac{\pi}{6}\qquad\qquad \text{Amplitude}=-\frac{\pi}{6}$$

3. Find the modulus and argument of $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$

Solution:

$$\text{Let } Z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$$

$$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$$

$$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{\left(\sqrt{3}\right)^2-4i^2}$$

$$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4} = \frac{7\sqrt{3}+7i}{7}$$

$$= \frac{7\left(\sqrt{3}+i\right)}{7}$$

$$= \sqrt{3}+i = r[\cos\theta + i\sin\theta]$$

$$r\cos\theta = \sqrt{3} = a, \quad r\sin\theta = 1 = b$$

$$\text{modulus} = r = \sqrt{a^2+b^2} = \sqrt{\left(\sqrt{3}\right)^2+i^2} = \sqrt{3+1} = \sqrt{4}$$

$$\text{Argument} = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \theta = \tan 30^\circ = \frac{\pi}{6}$$

4. Express $\frac{1+2i}{1-3i}$ in the form r(cosθ+isinθ)

Solution:

$$\text{Let } Z = \frac{1+2i}{1-3i}$$

$$Z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1+3i+2i+6i^2}{1-9i^2} = \frac{1+5i-6}{1+9}$$

$$= \frac{-5+5i}{10} = \frac{5(-1+i)}{10} = \frac{1}{2}(-1+i)$$

$$= \frac{-1}{2} + \frac{1}{2}i = r[\cos\theta + i\sin\theta]$$

$$r\cos\theta = \frac{-1}{2} = a \quad r\sin\theta = \frac{1}{2} = b$$

$$\text{modulus} = r = \sqrt{a^2+b^2} \quad = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{a}{r} = \frac{-\frac{1}{2}}{\frac{1}{\sqrt{2}}} = -\frac{1}{2}\times\frac{c}{1} = -\frac{1}{\sqrt{2}}$$

$$\sin\theta = \frac{b}{r} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{2}\times\frac{\sqrt{2}}{1} = \frac{1}{\sqrt{2}}$$

$$\cos 135^\circ = \cos(90+45) = -\sin 45 = -\frac{1}{\sqrt{2}}$$

$$\sin 135^\circ = \sin(90+45) = \cos 45 = \frac{1}{\sqrt{2}}$$

$$\text{Argument} \quad \theta = 135^\circ = \frac{3\pi}{4}$$

$$\therefore \frac{1+2i}{1-3i} = \frac{1}{\sqrt{2}}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

5. Find the modulus and amplitude of $\frac{5-i}{2-3i}$

Solution:

$$\text{Let } Z = \frac{5-i}{2-3i}$$

$$= \frac{5-i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{10+15i-2i-3i^2}{4-9i^2} = \frac{10+13i+3}{4+9}$$

$$= \frac{13+13i}{4+9} = \frac{13(1+i)}{13}$$

$$1+i = r[\cos\theta + i\sin\theta]$$

$$r\cos\theta = 1 = a \qquad r\sin\theta = 1 = b$$

$$\text{modulus} = r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\text{Argument } = \theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1}{1}\right)$$

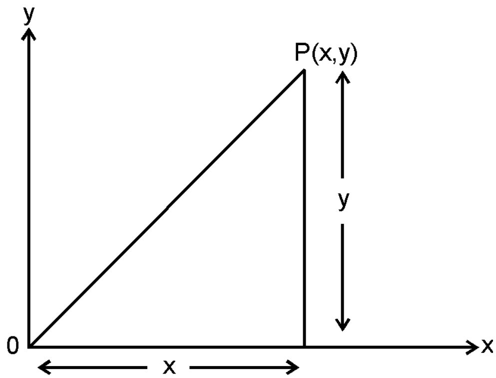
$$= \tan^{-1}(1)$$

$$\theta = \frac{\pi}{4}$$

1.3. ARGAND PLANE

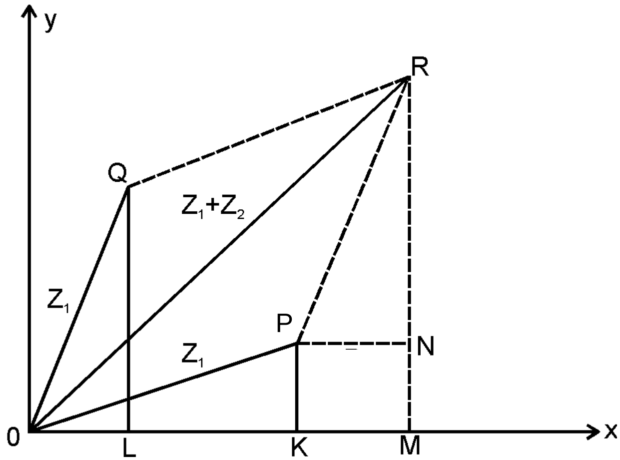
We have discussed some properties of a complex number in the previous section. There we have defined addition, subtraction, multiplication and division of complex numbers. Here we shall give the geometrical representation of a complex number.

Let Z=(x,y)=x+iy be a complex number.



We can represent a complex number Z=x+iy by a point P whose coordinates are (x,y) referred to the rectangular axes OX and OY known as real and imaginary axes respectively. The point P is known as image of the complex number Z and the complex number Z is known as the complex coordinates of the point P. The plane whose points are represented by the complex numbers is called the argand plane or argand diagram or complex plane or a Gaussian plane.

Sum of Two Complex Numbers:



Let OX is the real axis and OY is the imaginary axis. Let $Z_1=x_1+iy_1$ and $Z_2=x_2+iy_2$ be two complex numbers represented by the points P and Q on the Argand diagram. Complete the Parallelogram OPRQ. Draw PK, RM, QL perpendiculars to ox. Also draw PN ⊥r RM

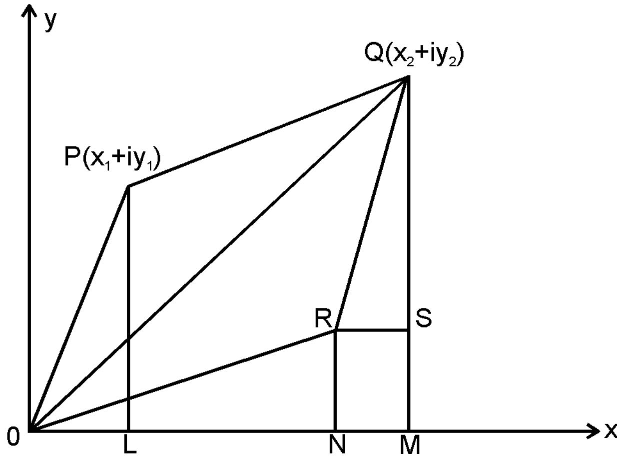
OM=OK+KM=OK+OL=x_1+x_2 and RM=MN+NR=KP+LQ=y_1+y_2

∴ The coordinates of R are (x_1+x_2, y_1+y_2) and it represent the complex number.

$(x_1+x_2)+i(y_1+y_2) = (x_1+iy_1)+(x_2+iy_2)$

Thus the sum of two complex numbers is represented by the extremity of the diagonal of the parallelogram formed by OP (Z_1) OQ (Z_2) as adjacent sides $\left|Z_1+Z_2\right|$=OR and arg (Z_1+Z_2)=∠ROM

Subtraction of Complex Numbers:



Let OX be the real axis and OY be the imaginary axis. Let P and Q represent the complex numbers (x_1+iy_1) and (x_2+iy_2). Draw PL, QM, RN ⊥^{rs} to OX and RS⊥ QM. Join OP and OQ and complete the parallelogram OPQR having OP and OQ as a side and a diagonal. Then R represents the difference of two complex numbers.

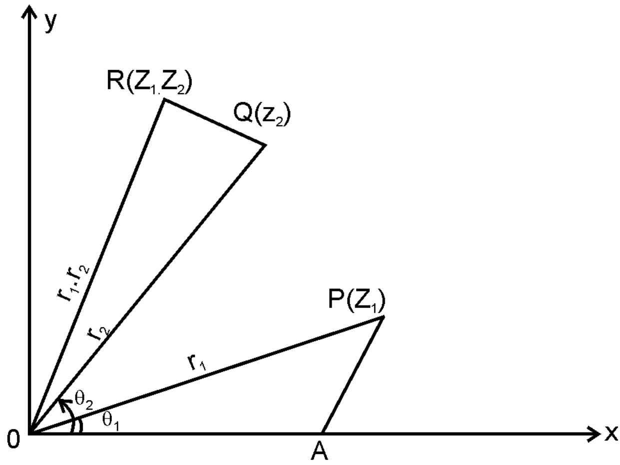
Now ΔOLP =ΔRSQ

OL=RS, PL=QS, ON=OM-NM=OM-OL=x_2-x_1

RN=SM=QM-QS=QM-PL=y_2-y_1

∴R represents the complex number (x_2-x_1, y_2-y_1)

Multiplication of Two Complex Numbers:



Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be two complex numbers. Now we know that a complex numbers z can be put in the form

$z=r\ (\cos\theta+i\sin\theta)$ where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Let $Z_1=r_1[\cos\theta_1+i\sin\theta_1]$

$Z_2=r_2[\cos\theta_2+i\sin\theta_2]$

Then $Z_1Z_2=r_1r_2\ [\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$

So that r_1r_2 is the modulus and Q_1+Q_2 is the argument of Z_1Z_2

Let $OP=r_1$ and $OQ=r_2$

$\angle POX=\theta_1 \qquad \angle QOX=\theta_2$

Let A be a point on the real axis such that OA=1. Join PA. Now draw the triangle OQR similar to the ΔOAP. Then the point Ris the required point. Now in the ΔOAP and ΔOQR

$$\frac{OR}{OQ}=\frac{OP}{OA}\Rightarrow\frac{OR}{r_2}=\frac{r_1}{1}\Rightarrow\ OR\ =r_1r_2$$

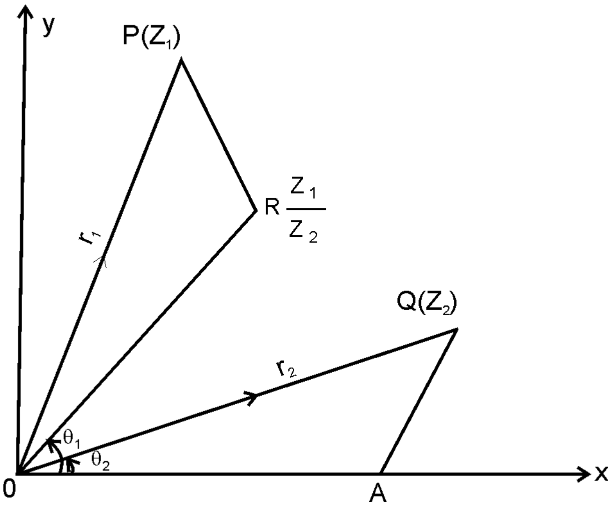
and $\quad = \angle XOR = \angle XOR + \angle QOR$

$$= \angle XOQ + \angle AOP$$

$$= \theta_2 + \theta_1$$

Hence R is the required point.

Division of Complex Numbers:



Let $Z_1 = r_1$(cosθ$_1$+isinθ$_2$) and Z_2=r_2 (cosθ$_2$+1sinθ$_2$) be the two complex numbers, then their quotient $R\frac{Z_1}{Z_2}$ is $\frac{Z_1}{Z_2} = \frac{r_1}{r_2}$[cos (θ$_1$-θ$_2$) + isin (θ$_1$-θ$_2$)]. Thus the modulus and argument of the complex number $\frac{Z_1}{Z_2}$ is $\frac{r_1}{r_2}$ and θ$_1$-θ$_2$ respectively. Now the point R representing the complex number $\frac{Z_1}{Z_2}$ is the point in the argand plane with polar co-ordinates $(\frac{r_1}{r_2}, \theta_1 - \theta_2)$. Let A be a point on the real axis such that OA=1. Let P and Q represent the complex numbers Z_1 and Z_2 .Now construct the triangle OPR similar to ΔOQA, then we have

$$\frac{OP}{OR} = \frac{OQ}{OA} \Rightarrow \frac{r_1}{OR} = \frac{r_2}{1}$$

$$OR = \frac{r_1}{r_2}$$

Also $\angle XOR = \angle XOP - \angle ROP \qquad = \theta_1 - \theta_2$

Thus R is the required point.

Note:

If $Z_1 = x_1 + iy_1$, and $Z_2 = x_2 + iy_2$ then

$Z_1 - Z_2 = (x_1 - x_2) + i(y_1 - y_2)$ and

$|Z_1 - Z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ clearly $|Z_1 - Z_2|$ is the distance between the point represented by the complex numbers.

1.3 WORKED EXAMPLES

PART - A

1. Find the distance between the points represented by the complex numbers (2+i) and 1-2i

Solution:

Let A (2+i) and 1-2i

$$\begin{aligned} AB &= |(2+i)-(1-2i)| \\ &= |2+i-1+2i| \\ &= |1+3i| \\ &= \sqrt{1^2+3^2} = \sqrt{1+9} = \sqrt{10} \text{ units.} \end{aligned}$$

PART - B

1. Prove that points in the Argand plane representing the complex numbers 3+7i, 6+5i and 15-i are collinear.

Solution:

Let A(3+7i), B(6+5i), C(15-i) be the point representing the complex numbers in the Argand plane.

$$AB = \left| (3+7i) - (6+5i) \right| = \left| 3+7i-6-5i \right|$$

$$= \left| -3+2i \right|$$

$$= \sqrt{(-3)^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$$

$$BC = \left| (6+5i) - (15-i) \right| = \left| 6+5i-15+i \right|$$

$$= \left| -9+6i \right|$$

$$= \sqrt{(-9)^2 + 6^2} = \sqrt{81+36} = \sqrt{117} = \sqrt{9 \times 13} = 3\sqrt{13}$$

$$AC = \left| (3+7i) - (15-i) \right| = \left| 3+7i-15+i \right|$$

$$= \left| -12+8i \right|$$

$$= \sqrt{(-12)^2 + 8^2} = \sqrt{144+64} = \sqrt{208} = \sqrt{16 \times 13} = 4\sqrt{13}$$

$$\therefore AB + BC = \sqrt{13} + 3\sqrt{13}$$

$$= 4\sqrt{13}$$

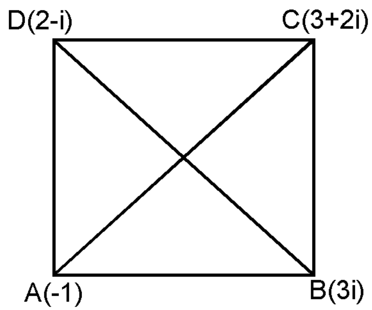
$$AB + BC = AC$$

$\therefore$ The given points are collinear.

2. Prove that the complex numbers -1,3i, 3+2i and 2-I are the vertices of a square in the argand plane.

Solution:

Let A(-1), B(3i), C(3+2i), D(2-i) be the four points in the argand plane.



$$AB = |-1-3i|$$

$$= \sqrt{(-1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

$$BC = |3i-(3+2i)| = |3i-3-2i|$$

$$= |-3+i| = \sqrt{(-3)^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

$$CD = |(3+2i)-(2-i)| = |3+2i-2+i| = |1+3i|$$

$$= \sqrt{1^2+3^2} = \sqrt{1+9} = \sqrt{10}$$

$$DA = |(2-i)-(-1)| = |2-i+1| = |3-i|$$

$$= \sqrt{3^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}$$

$$\therefore AB = BC = CD = DA = \sqrt{10}$$

$\therefore$ All four sides are equal.

Also diagonal

$$AC = |-1-(3+2i)| = |-1-3-2i| = |-4-2i|$$

$$= \sqrt{(-4)^2+(-2)^2} = \sqrt{16+4} = \sqrt{20}$$

$$BD = |3i-(2-i)| = |3i-2+i| = |-2+4i|$$

$$= \sqrt{(-2)^2+4^2} = \sqrt{4+16} = \sqrt{20}$$

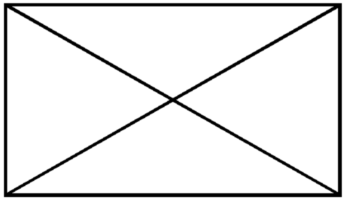
$\therefore$ AC=BD

$\therefore$ The given points form a square.

3. Show that the points representing the complex numbers 2-2i, 8+4i, 5+7i and -1+i form the vertices of a rectangle.

Solution:

Let A(2-2i), B(8+4i), C(5+7i) and D(-1+i) be the points representing the complex numbers in the argand diagram.



$$AB = |(2-2i)-(8+4i)| = |2-2i-8-4i|$$
$$= |-6-6i| = \sqrt{(-6)^2+(-6)^2}$$
$$= \sqrt{36+36} = \sqrt{72}$$

$$BC = |(8+4i)-(5+7i)| = |8+4i-5-7i| = |3-3i|$$
$$= \sqrt{3^2+(-3)^2} = \sqrt{9+9} = \sqrt{18}$$

$$CD = |(5+7i)-(-1+i)| = |5+7i+1-i| = |6+6i|$$
$$= \sqrt{6^2+6^2} = \sqrt{36+36} = \sqrt{72}$$

$$DA = |(-1+i)-(2-2i)| = |-1+i-2+2i| = |-3+3i|$$
$$= \sqrt{(-3)^2+3^2} = \sqrt{9+9} = \sqrt{18}$$

$\therefore$ AB=CD and BC=DA

Also diagonals

$$AC = |(2-2i)-(5+7i)| = |2-2i-5-7i| = |-3-9i|$$
$$= \sqrt{(-3)^2+(-9)^2} = \sqrt{9+81} = \sqrt{90}$$

$$BD = |(8+4i)-(-1+i)| = |8+4i+1-i| = |9+3i|$$
$$= \sqrt{9^2+3^2} = \sqrt{81+9} = \sqrt{90}$$

$\therefore$ AC=BD

$\therefore$ Opposite sides are equal and also diagonals are equal.

$\therefore$ The given four points form a rectangle.

4. Show that the complex numbers 9+I, 4+13i, -8+8i, and -3-4i form a rhombus.

Solution:

Let A(9+i), B(4+13i), c(-8+8i) and D(-3-4i) be the points representing the complex numbers in the Argand plane.

$$AB = \left|(9+i)-(4+13i)\right| = \left|9+i-4-13i\right| = \left|5-12i\right|$$

$$= \sqrt{5^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$

$$BC = \left|(4+13i)-(-8+8i)\right| = \left|4+13i+8-8i\right| = \left|12+5i\right|$$

$$= \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13$$

$$CD = \left|(-8+8i)-(-3-4i)\right| = \left|-8+8i+3+4i\right| = \left|-5+12i\right|$$

$$= \sqrt{(-5)^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$$

$$AD = \left|(9+i)-(-3-4i)\right| = \left|9+i+3+4i\right| = \left|12+5i\right|$$

$$= \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13$$

$\therefore$ AB=BC=CD=AD=13

i.e. All the four sides are equal.

$\therefore$ ABCD is a rhombus.

5. Prove that the complex numbers 1-2i, -1+4i, 5+8i, 7+2i form a parallelogram.

Solution:

Let A(1-2i), B(-1+4i), C(5+8i) and D(7+2i) be the points representing the complex numbers in the argand plane.

$$\text{Mid point of AC} = \frac{1-2i+5+8i}{2} = \frac{6+6i}{2} = 3+3i$$

$$\text{Mid point of BD} = \frac{-1+4i+7+2i}{2} = \frac{6+6i}{2} = 3+3i$$

Since the midpoint of AC and BD are the same, the diagonals AC and BD bisect each other.

$\therefore$ ABCD is a parallelogram.

6. Show that the points representing the complex numbers 7+9i, -3+3i form a right angled isosceles triangle in the argand diagram.

Solution:

Let A (7+9i),(-3+7i) and B(-3+7i) and C(3+3i) be the points representing the complex number in the Argand plane

$$AB = |(7+9i)-(-3+7i)| = |7+9i+3-7i| = |10+2i|$$

$$= \sqrt{10^2+2^2} = \sqrt{100+4} = 104$$

$$BC = |(-3+7i)-(3+3i)| = |-3+7i-3-3i| = |-6+4i|$$

$$= \sqrt{(-6)^2+4^2} = \sqrt{36+16} = \sqrt{52}$$

$$CA = |(3+3i)-(7+9i)| = |3+3i-7-9i| = |-4-6i|$$

$$= \sqrt{(-4)^2+(-6)^2} = \sqrt{16+36} = \sqrt{52}$$

Here $BC = CA = \sqrt{52}$

and $BC^2 + CA^2 = 52 + 52 = 104$

$AB^2 = 104$

$$\therefore BC^2 + CA^2 = AB^2$$

$\therefore$ ABC is a right angled triangle.

Hence Δ ABC is a right angled isosceles triangle.

EXERCISE
PART – A

1. If z_1=(5,2) and z_2=(-2,3) find $3z_1+5z_2$
2. If z_1=(-1,0) and z_2=(4,5) find $7z_1-2z_2$
3. If z_1=(6,4) and z_2=(-5,2) find z_1z_2
4. What is the value of i^3+i^5

5. What is the value of $i^8 + i^4$

6. If $z = \bar{z}$ what is the nature of Z?

7. Write down the complex conjugate of

(i) c + id (ii) 2i – 5 (iii) 4 - 6i (iv) 8i

(v) 7 + 9i (vi) $\frac{1}{4-3i}$

8. Find the modulus of the following

(i) 2-i (ii) $\sqrt{3}+2i$ (iii)-1-5i

(iv) 1+i (iv) -i (v)i

9. What is the amplitude of the following

(i) $\sqrt{3}+i$ (ii) 5 (iii) -1-I (iv) -1-i$\sqrt{3}$

10. If $z_1 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$, $z_2 = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

a) What is amplitude of the following complex numbers?

(i) $z_1 z_2$ (ii) $\frac{z_2}{z_1}$

b) What is the modulus of the complex numbers?

(iii) $z_1 z_2$ (iv) $\frac{z_1}{z_2}$

11. Find $z_1 z_2$ and $\frac{z_1}{z_2}$ if

(i) z_1 = 3(cos10°+isin10°)

z_2 = 4(cos20°+isin20°)

(ii) z_1 =10(cos40°+isin40°)

z_2 = 5(cos70°+isin70°)

12. Find the value of

(cos20°+isin20°) (cos30°+isin30°) (cos40°+isin40°)

13. Prove that $\frac{1+i}{1-i} = i$

14. Prove that $\frac{1-i}{1+i} = -i$

15. Find the distance between the complex numbers

(i) 2+3i and 3-2i (ii) 2+3i and 3-i

(iii) 1+i and 3-2i (iv) 2-3i and 5+7i

PART-B

1. Find the real and imaginary parts of

(a) $\frac{1+i}{1-i}$ (b) $\frac{1}{5+4i}$ (c) $\frac{1-i}{1+i}$ (d) $\frac{2}{3+i}$

(e) $\frac{1}{1+i}$ (f) $\frac{1}{4-5i}$ (g) $\frac{3}{2+i}$ (h) $\frac{3}{3+4i}$

(i) $\frac{5}{4-2i}$ (j) $\frac{1}{\alpha+i\beta}$

2. Find the real and imaginary part of

(a) $\frac{1}{3+4i} - \frac{5}{4-3i}$ (b) $\frac{3-2i}{5+4i} + \frac{1-3i}{4-5i}$

(c) $\frac{2-i}{7-4i} + \frac{1-i}{3+2i}$ (d) $\frac{3}{3+4i} + \frac{1}{5-2i}$

(e) $\left(\frac{1-I}{1+I}\right)^2$ (f) $\frac{3+i}{2-5i} + \frac{3-4i}{5+2i}$

(g) $\frac{1}{2+3i} + \frac{1}{1-i}$ (h) $\alpha + i\beta$

(i) $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ (j) $\frac{2+i}{(1+i)^2}$

(k) $\frac{4}{3+2i} + \frac{2}{5-4i}$ (l) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$

(m) $\frac{i-4}{3-2i} + \frac{1+4i}{2-3i}$ (n) $\frac{1}{1+\cos\theta + i\sin\theta}$

3. Express in the form a + ib

(a) $\frac{(1+i)^2}{1-i}$ (b) $\frac{(2+i)^2}{(3+2i)}$

(c) $\frac{3}{4+3i}+\frac{i}{3-4i}$ (d) $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3$

(e) $\frac{7-5i}{(2+3i)^2}$ (f) $\frac{(1+i)(1-2i)}{(1+3i)}$

(g) $\frac{2+3i}{1-i}$ (h) $\frac{(1+i)(1+2i)}{1+4i}$

(i) $(1+2i)(1+i)^2$ (j) $\frac{(1+i)(3+i)^2}{(2-i)^2}$ (k) $\frac{(2+3i)^2}{(1-2i)^2}$

4. Find the conjugate of the following

(a) $\frac{(1+i)^2}{1-i}$ (b) $\frac{(1+i)(2+i)}{(3+2i)}$ (c) $\frac{1+2i}{1-3i}$

(d) $\frac{1-i}{3+2i}$ (e) $\frac{3+i}{2+5i}$ (f) $\frac{1-i}{(2+i)^2}$

(g) $\frac{1}{(1+i)^2}-\frac{1}{(1-i)^2}$ (h) $\frac{(1+i)^2+(1-i)^2}{(1+i)^2-(1-i)^2}$

5. Find the modulus and argument of

(a) $1+i$ (b) i

(c) $1+\sqrt{3}i$ (d) $-1+\sqrt{3}i$

(e) $-1-\sqrt{3}i$ (f) $3+4i$

(g) $2-i$ (h) $1+i\tan\theta$

6. Find the modulus and amplitude (or) argument (or) principal value of the following

(a) $\frac{(1+i)(2+i)}{(3-i)}$ (b) $\frac{2-i}{3+7i}$

(c) $\frac{1-i}{1+i}$ (d) $(1+i)(3+4i)$

(e) $\dfrac{1+\sqrt{3}i}{1+i}$ (f) $\dfrac{1}{1+\cos\theta+i\sin\theta}$

(g) $\dfrac{i-3}{i-1}$ (h) $\dfrac{1+i}{\sqrt{3}+i}$

(i) $\dfrac{5-i}{2-3i}$ (j) $\dfrac{1+i\sqrt{3}}{1-i}$ (k) $\dfrac{(1+i)(1+2i)}{1+3i}$

7. Find the argument of the sum of the complex numbers (1,0) and (0,1)

8. Find the product of $3\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)$ and $4\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$

9. Find the product of $5\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$ and $3\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$

10. Simplify:

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

11. Sow that the following complex numbers are collinear

(a) 4+2i, 7+5i, 9+7i (b) 1+3i, 5+i, 3+2i

(c) 1+3i, 2+7i, -2-9i (d) 5+8i, 13+20i, 19+29i

(e) 3+7i, 6+5i, 15-i

12. Prove that the following complex numbers form a square

(a) 9+i, 4+13i, -8+8i, -3-4i

(b) 3+2i, 5+4i, 3+6i, 1+4i

(c) 2+i, 4+3i, 2+5i, 3i

(d) 4+5i, 1+2i, 4-i, 7+2i

(e) -1, 3i, 3+2i, 2-i

(f) –i, 2+i, 3i, -2+i

13. Show that the following complex numbers form a rectangle

(a) 1+2i, -2+5i, 7i, 3+4i

(b) 4+3i, 12+9i, 15+5i, 7-i

(c) 1+i, 3+5i, 4+4i, 2i

(d) 2-2i, 8+4i, 5+7i, -1+i

14. Prove that the following complex numbers form a rhombus

(a) 2+i, 4+3i, 2+5i, 3i

(b) 3+4i, 9+8i, 5+2i, -1+2i

(c) 8+5i, 16+11i, 10+3i, 2-3i

(d) 6+3i, 4+5i, 6+3i, 8+i

15. Prove that the following complex numbers from a parallelogram

(a) 3+i, 2+2i, -2+i, -1

(b) 2-2i, 8+4i, 5+7i, -1+i

(c) 7, 4+3i, -2+5i, 1+2i

(d) -3+3i, -2i, 2+6i, 5+i

(e) 1-2i, -1+4i, 5+8i, 7+2i

(f) 1, 4+3i, -2-i, 1+2i

16. Prove that the points represented by the following complex numbers form an equilateral triangle.

(i) $2+2i,\ -2-2i,\ -2\sqrt{3}+2\sqrt{3}i$

(ii) $1,\ \frac{1}{2}\left(-1+\sqrt{3}i\right),\ \frac{1}{2}\left(-1-\sqrt{3}i\right)$

(iii) $\sqrt{3}-\sqrt{3}i,\ \ -1-i,\ 1+i$

17. Prove that the following complex numbers form a right-angled triangle

(i) 2-3i, -6-7i, -8-3i

(ii) -3-4i, 2+6i, -6+10i

ANSWERS
PART – A

(1) (5,21) (2) (-15,-10) (3) (-38,-8)

(4) 0 (5) 2 (6) z is a Real Number

(7) (i) c-id (ii) -2i-5 (iii) 4+6i (iv) -8i

(v) 7-9i (vi) $\frac{4}{25}-\frac{3i}{25}$

(8) (i) $\sqrt{5}$ (ii) $\sqrt{7}$ (iii) $\sqrt{26}$

(iv) $\sqrt{2}$ (v) 1 (vi) 1

(9) (i) $30°(or)\frac{\pi}{6}$ (ii) 0 (iii) $\frac{\pi}{4}(or)45°$ (iv) $\frac{\pi}{3}(or)60°$

(10) (a) (i) $\frac{\pi}{2}$ (ii) $\frac{-\pi}{4}$

(b) (i) 8i (ii) $\frac{1}{2\sqrt{2}}(1-i)$

(11) (i) $6(\sqrt{3}+i), \frac{3}{4}(\cos 10° - i\sin 10)$

(ii) $50(\cos 110 + i\sin 110°), \sqrt{3}-i$

(12) i

(13.) (i) $\sqrt{26}$ (ii) $\sqrt{17}$ (iii) $\sqrt{13}$ (iv) $\sqrt{109}$

ANSWERS
PART - B

1. (a) 0,1 (b) $\frac{5}{41}, \frac{-4}{41}$ (c) 0,-1

(d) $\frac{3}{5},\frac{-1}{5}$ (e) $\frac{1}{2},\frac{-1}{2}$ (f) $\frac{4}{41},\frac{5}{41}$

(g) $\frac{6}{5},\frac{-3}{5}$ (h) $\frac{9}{25},\frac{-12}{25}$ (i) $1, \frac{1}{2}$

(j) $\frac{\alpha}{\alpha^2+\beta^2},\frac{\beta}{\alpha^2+\beta^2}$

2. (a) $\frac{-17}{25},\frac{-19}{25}$ (b) $\frac{26}{41},\frac{-29}{41}$

(c) $\frac{23}{65},\frac{-24}{65}$ (d) $\frac{386}{725},\frac{-298}{725}$

(e) -1,0 (f) $\frac{8}{29},\frac{-9}{29}$

(g) $\frac{17}{26},\frac{7}{26}$ (h) $\frac{193}{41},\frac{149}{41}$

(i) 0,2 (j) $\frac{1}{2},-1$

(k) $\frac{622}{533},\frac{-224}{533}$ (l) $\sqrt{3},1$

(m) $\frac{-312}{169},\frac{78}{169}$ (n) $\frac{1}{2},\frac{-1}{2}\tan\frac{\theta}{2}$

3. (a) -1+I (b) $\frac{17}{13}+\frac{6}{13}i$ (c) $\frac{8}{25}-\frac{6}{25}i$

(d) -2i (e) $\frac{-95}{169}-\frac{59}{169}i$ (f) -i

(g) $\frac{-1}{2}+\frac{5}{2}i$ (h) $\frac{11}{17}+\frac{7}{17}i$ (i) -4+2i

(j) -2+2i (k) $\frac{-33}{25}-\frac{56}{25}i$

4. (a) -1-i (b) $\frac{9}{13}-\frac{7i}{13}$ (c) $\frac{-1}{2}-\frac{1}{2}i$

(d) $\frac{1}{13}+\frac{5}{13}i$ (e) $\frac{11}{29}-\frac{13}{29}i$ (f) $\frac{-1}{25}-\frac{7}{25}i$

(g) -i (h) 0

5. (a) $r=\sqrt{2},\ \theta=45^\circ$ (b) $r = 1,\ \theta = 90^\circ$

(c) $r = 2,\ \theta = 60^\circ$ (d) $r = 2,\ \theta = 120^\circ$

(e) $r = 2,\ \theta = -120^\circ$ (f) $r=5,\ \ \theta=\tan^{-1}\left(\frac{4}{3}\right)$

(g) $\sqrt{5},\tan^{-1}\left(\frac{1}{2}\right)$ (h) $r=\sec\theta,\ \phi=\theta$

6. (a) $1,\frac{\pi}{2},$ (b) $\frac{\sqrt{290}}{58},\tan^{-1}(17)$ (c) $1,-\frac{\pi}{2}$

(d) $\sqrt{50},\tan^{-1}(-7),$ (e) $\sqrt{2},\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$

(f) $\frac{1}{2}\sec\frac{\theta}{2},\frac{\theta}{2}$ (g) $\sqrt{5},\tan^{-1}\left(\frac{1}{2}\right),$

(h) $\frac{1}{\sqrt{2}},\tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ (i) $\sqrt{2},\frac{\pi}{4}$

(j) $\sqrt{2},\tan^{-1}\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)$ (k) $1,\tan^{-1}\left(\frac{3}{4}\right)$

7. $\frac{\pi}{4}$ 8. 12i 9. -15 10. $\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$

UNIT – II

COMPLEX NUMBER-II

2.1 Demoivre's Theorem (statement only) –simple Problems

2.2 Demoivre's Theorem related Problems. Simple Problems

2.3 Finding the n^{th} roots of unity – Solving equation of the form $x^n \pm 1 = 0$ where $n \le 7$ Simple Problems.

2.1 DE-MOIVRE'S THEOREM

Statement: (i) If n is an integer (positive or negative)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) If n is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$

Examples:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$(\cos\theta + i\sin\theta)^{-2} = \cos 2\theta - i\sin 2\theta$$

$$(\cos 4\theta + i\sin 4\theta)^2 = \cos 8\theta + i\sin 8\theta$$

Note:

1 If $\cos\theta + i\sin\theta = z$

$$\frac{1}{z} = \frac{1}{(\cos\theta + i\sin\theta)} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$$

2. If $z = \cos\theta - i\sin\theta$ then $\dfrac{1}{z} = \dfrac{1}{\cos\theta - i\sin\theta} = \cos\theta + i\sin\theta$

3. If $z = \cos\theta + i\sin\theta$ then

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

4. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

$z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$

(i) $z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\times r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

(ii) $z_1z_2z_3 = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)r_3(\cos\theta_3 + i\sin\theta_3)$

$$= r_1r_2r_3[\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)]$$

(iii) $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Euler's formula:

The values of e^x, cosx and sinx can also be given in the form of series as below.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots.$$

$$\left.\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots \end{aligned}\right\} \qquad \dots 1$$

$$\therefore e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \dots\dots\dots.$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} - \dots\dots.$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots..\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}\dots.\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \text{using} \quad (1)$$

This is called Euler's formula to represent a complex number.

2.1 WORKED EXAMPLES
PART - A

1) If $z = \cos 30^\circ + i\sin 30^\circ$ what is the value of z^3

Solution:

$$z^3 = (\cos 30^\circ + i\sin 30^\circ)^3$$

$$= \cos 90^\circ + i\sin 90^\circ = 0 + i(1) = i$$

2) If $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ what is the value of z^8

Solution:

$$z^8 = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^8$$

$$= \cos 8\left(\frac{\pi}{2}\right) + i\sin 8\left(\frac{\pi}{2}\right)$$

$$= \cos 4\pi + i\sin 4\pi = 1 + i(0) = 1$$

3) If z= $\cos 90^\circ + i\sin 90^\circ$ what is the value of $z^{\frac{2}{3}}$

Solution:

$$z^{\frac{2}{3}} = (\cos 90 + i\sin 90)^{\frac{2}{3}} = \cos\frac{2}{3}(90^\circ) + i\sin\frac{2}{3}(90^\circ)$$

$$= \cos 60^\circ + i\sin 60^\circ = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

4) If $z = \cos 45^\circ - i\sin 45^\circ$ what is the value of $\frac{1}{z}$

Solution:

$$\frac{1}{z} = \frac{1}{\cos 45^\circ - i\sin 45^\circ}$$

$$= \cos 45^\circ + i\sin 45^\circ = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

5) If $\frac{1}{z} = \cos 60^\circ + i\sin 60^\circ$ what is the value of z

Solution:

$$z = \frac{1}{\left(\frac{1}{z}\right)} = \frac{1}{\cos 60^\circ + i\sin 60^\circ} = (\cos 60^\circ + i\sin 60^\circ)^{-1}$$

$$= \cos 60^\circ - i\sin 60^\circ$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

6) Find the value of $\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta + i\sin 3\theta}$

Solution:

$$\frac{\cos 5\theta + i\sin 5\theta}{\cos 3\theta + i\sin 3\theta} = \cos(5\theta - 3\theta) + i\sin(5\theta - 3\theta)$$

$$= \cos 2\theta + i\sin 2\theta$$

7) Find the product of $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$ and $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

Solution:

Product of two complex numbers

$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{6} + \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{6} + \frac{5\pi}{6}\right)$$

$$= \cos\pi + i\sin\pi$$

$$= -1 + i(0) = -1$$

8) If $x = \cos\theta + i\sin\theta$, Find $x + \frac{1}{x}$

Solution:

$$x = \cos\theta + i\sin\theta$$

$$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

9) If $x = \cos\alpha + i\sin\alpha, \qquad y = \cos\beta + i\sin\beta$, Find xy

Solution:

$$\begin{aligned} xy &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta). \end{aligned}$$

10) Simplify: $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Solution:

$$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{3\pi + 4\pi + 2\pi}{12}\right) + i\sin\left(\frac{3\pi + 4\pi + 2\pi}{12}\right)$$

$$= \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$$

$$= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

PART - B

1) Prove that $(\text{Sin}\,\theta + \cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

Solution:

$$(\text{Sin}\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$$

2) Simplify using De Moivre's theorem

$$\frac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 (\cos 5\theta + i\sin 5\theta)^{-4}}$$

Solution:

$$\frac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 (\cos 5\theta + i\sin 5\theta)^{-4}}$$

$$= \frac{\left[(\cos\theta + i\sin\theta)^3\right]^4 \left[(\cos\theta + i\sin\theta)^{-4}\right]^5}{\left[(\cos\theta + i\sin\theta)^4\right]^3 \left[(\cos\theta + i\sin\theta)^5\right]^{-4}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{12} (\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{12} (\cos\theta + i\sin\theta)^{-20}}$$

$$= (\cos\theta + i\sin\theta)^{12-20-12+20}$$

$$= (\cos\theta + i\sin\theta)^0 = 1$$

3) Simplify : $\dfrac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-6} (\cos\theta + i\sin\theta)^8}$

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 3\theta - i\sin 3\theta)^{-3}}{(\cos 4\theta + i\sin 4\theta)^{-6} (\cos\theta + i\sin\theta)^8}$$

$$= \frac{\left[(\cos\theta + i\sin\theta)^2\right]^3 \left[(\cos\theta + i\sin\theta)^{-3}\right]^{-3}}{\left[(\cos\theta + i\sin\theta)^4\right]^{-6} (\cos\theta + i\sin\theta)^8}$$

$$= \frac{(\cos\theta + i\sin\theta)^6 (\cos\theta + i\sin\theta)^9}{(\cos\theta + i\sin\theta)^{-24} (\cos\theta + i\sin\theta)^8}$$

$$= (\cos\theta + i\sin\theta)^{6+9+24-8}$$

$$= (\cos\theta + i\sin\theta)^{31}$$

$$= \cos 31\theta + i\sin 31\theta$$

4) Find the value of

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta} \text{ when } \theta = \frac{\pi}{9}$$

Solution:

$$\frac{(\cos 2\theta + i\sin 2\theta)^3 (\cos 4\theta - i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$$

$$= \frac{\left[(\cos\theta + i\sin\theta)^2\right]^3 \left[(\cos\theta + i\sin\theta)^{-4}\right]^3}{(\cos\theta + i\sin\theta)^3}$$

$$= \frac{(\cos\theta + i\sin\theta)^6 \ (\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^3}$$

$$= (\cos\theta + i\sin\theta)^{6-12-3}$$

$$= (\cos\theta + i\sin\theta)^{-9} = \cos 9\theta - i\sin 9\theta$$

when $\theta = \frac{\pi}{9}$

$$= \cos 9\left(\frac{\pi}{9}\right) - i\sin 9\left(\frac{\pi}{9}\right)$$

$$= \text{Cos}\pi - i\sin\pi$$

$$= -1 - i(0) \qquad = -1$$

5) Prove that $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$

Solution:

$$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \frac{[i(\cos\theta + i\sin\theta)]^4}{[i(\sin\theta + i\cos\theta)]^4}$$

$$= \frac{(i)^4(\cos\theta + i\sin\theta)^4}{(i\sin\theta + i^2\cos\theta)^4}$$

$$= \frac{1(\cos\theta + i\sin\theta)^4}{(i\sin\theta - \cos\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{[-(\cos\theta - i\sin\theta)]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{\left[(\cos\theta + i\sin\theta^{-1})\right]^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos\theta + i\sin\theta)^{4+4} = (\cos\theta + i\sin\theta)^8$$

$$= \cos 8\theta + i\sin 8\theta$$

6) Prove that $\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \cos n\theta + i\sin n\theta$

Solution:

Let $z = \cos\theta + i\sin\theta$ and

$$\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

$$\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$= \left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$== \left[\frac{z(1+z)}{(z+1)}\right]^n$$

$$= z^n = (\cos\theta + i\sin\theta)^n$$

$$= \cos n\theta + i\sin n\theta$$

7) Prove that $\dfrac{1-\cos 2\theta - i\sin 2\theta}{1+\cos 2\theta + i\sin 2\theta} = -i\tan\theta$

Solution:

$$\frac{1-\cos 2\theta - i\sin 2\theta}{1+\cos 2\theta + i\sin 2\theta} = \frac{2\sin^2\theta - i2\sin\theta\cos\theta}{2\cos^2\theta + i2\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta(\sin\theta - i\cos\theta)}{2\cos\theta(\cos\theta + i\sin\theta)}$$

$$= \tan\theta\frac{(\sin\theta - i\cos\theta)\ x\,i}{(\cos\theta + i\sin\theta)\ x\,ii}$$

$$= \frac{\tan\theta\left(i\sin\theta - i^2\cos\theta\right)}{i(\cos\theta + i\sin\theta)}$$

$$= \frac{\tan\theta[\cos\theta + i\sin\theta]}{i(\cos\theta + i\sin\theta)}$$

$$= \frac{\tan\theta[\cos\theta + i\sin\theta]}{i(\cos\theta + i\sin\theta)} = -i\tan\theta$$

2.2 DE-MOIVRE'S THEOREM RELATED PROBLEMS

PART - B

1) If a = $\cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ prove that

(i) $\cos(\alpha+\beta) = \dfrac{1}{2}\left[ab + \dfrac{1}{ab}\right]$

(ii) $\sin(\alpha-\beta) = \dfrac{1}{2i}\left[\dfrac{a}{b} - \dfrac{b}{a}\right]$

Solution:

(i) If $a = \cos\alpha + i\sin\alpha$, and $b = \cos\beta + i\sin\beta$

$$ab = (\cos\alpha + i\sin\alpha)\,(\cos\beta + i\sin\beta)$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$= \frac{1}{ab} = \frac{1}{\cos(\alpha+\beta) + i\sin(\alpha+\beta)} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$ab + \frac{1}{ab} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$= 2\cos(\alpha+\beta)$$

$$\cos(\alpha+\beta) = \frac{1}{2}\left[ab + \frac{1}{ab}\right]$$

$$\frac{a}{b} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$\frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{\cos(\alpha-\beta) + i\sin(\alpha-\beta)} = \cos(\alpha-\beta) - i\sin(\alpha-\beta)$$

(ii) $$\frac{a}{b} - \frac{b}{a} = \cos(\alpha-\beta) + i\sin(\alpha-\beta) - \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$= 2i\sin(\alpha-\beta)$$

$$\therefore \sin(\alpha-\beta) = \frac{1}{2i}\left(\frac{a}{b} - \frac{b}{a}\right)$$

2) If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$ prove that

(i) $\sqrt{ab} + \frac{1}{\sqrt{ab}} = 2\cos\left(\frac{x+y}{2}\right)$

(ii) $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos\left(\frac{x-y}{2}\right)$

Solution:

When $a = \cos x + i\sin x$, $b = \cos y + i\sin y$

(i) $ab = (\cos x + i\sin x)(\cos y + i\sin y)$

$$= \cos(x+y) + i\sin(x+y)$$

$$= \sqrt{ab} = (ab)^{\frac{1}{2}} = [\cos(x+y) + i\sin(x+y)]^{\frac{1}{2}}$$

$$= \cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)$$

$$\frac{1}{\sqrt{ab}} = \frac{1}{\cos\left(\frac{x+y}{2}\right) + i\sin\left(\frac{x+y}{2}\right)} = \cos\left(\frac{x+y}{2}\right) - i\sin\left(\frac{x+y}{2}\right)$$

$$\sqrt{ab}+\frac{1}{\sqrt{ab}}=\cos\left(\frac{x+y}{2}\right)+i\sin\left(\frac{x+y}{2}\right)+\cos\left(\frac{x+y}{2}\right)-i\sin\left(\frac{x+y}{2}\right)$$

$$\sqrt{ab}+\frac{1}{\sqrt{ab}}=2\cos\left(\frac{x+y}{2}\right)$$

$$\frac{a}{b}=\frac{\cos x+i\sin x}{\cos y+i\sin y}=\cos(x-y)+i\sin(x-y)$$

$$\sqrt{\frac{a}{b}}=\left(\frac{a}{b}\right)^{\frac{1}{2}}=[\cos(x-y)+i\sin(x-y)]^{\frac{1}{2}}=\cos\left(\frac{x-y}{2}\right)+i\sin\left(\frac{x-y}{2}\right)$$

$$\sqrt{\frac{b}{a}}=\frac{1}{\sqrt{\frac{a}{b}}}=\frac{1}{\cos\left(\frac{x-y}{2}\right)+i\sin\left(\frac{x-y}{2}\right)}=\cos\left(\frac{x-y}{2}\right)-i\sin\left(\frac{x-y}{2}\right)$$

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\cos\left(\frac{x-y}{2}\right)+i\sin\left(\frac{x-y}{2}\right)+\cos\left(\frac{x-y}{2}\right)-i\sin\left(\frac{x-y}{2}\right)$$

$$=2\cos\left(\frac{x-y}{2}\right)$$

3) If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\upsilon+i\sin\upsilon$ and if $x+y+z=0$ then prove that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$

Solution:

given $x=\cos\alpha+i\sin\alpha, y=\cos\beta+i\sin\beta,\ z=\cos\upsilon+i\sin\upsilon$

$$x+y+z=0$$

$$\therefore\ \cos\alpha+i\sin\alpha+\cos\beta+i\sin\beta+\cos\upsilon+i\sin\upsilon=0$$

$$\cos\alpha+\cos\beta+\cos\upsilon+i(\sin\alpha+\sin\beta+\sin\upsilon)=0+i(0)$$

Equating real and imaginary parts

$$\cos\alpha+\cos\beta+\cos\upsilon=0$$

$$\sin\alpha+\sin\beta+\sin\upsilon=0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{\cos\alpha+i\sin\alpha}+\frac{1}{\cos\chi+i\sin\beta}+\frac{1}{\cos\upsilon+i\sin\upsilon}$$

$$=\cos\alpha-i\sin\alpha+\cos\beta-i\sin\beta+\cos\upsilon-i\sin\upsilon$$

$$=\cos\alpha+\cos\beta+\cos\upsilon-i(\sin\alpha+\sin\beta+\sin\upsilon)$$

$$=0-i(0)=0$$

4) Show that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

Solution:

$\text{Let } 1+i=r(\cos\theta+i\sin\theta)$

Hence a=1, b=1

$$r=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$$

$$\theta=\tan^{-1}\left(\frac{b}{a}\right)=\tan^{-1}\left(\frac{1}{1}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

$$\therefore 1+i=\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right] \qquad \ldots 1$$

Replace i by – i i n (1),we get

$$1-i=\sqrt{2}\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]$$

$$(1+i)^n+(1-i)^n=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n+\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^n$$

$$=\left[\sqrt{2}^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)\right]^n+\left[\sqrt{2}^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)\right]$$

$$=\left(\sqrt{2}^n\right)\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$$

$$=2^{\frac{n}{2}}\left[2\cos\frac{n\pi}{4}\right]$$

$$=2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

$$=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$$

5) Show that $\left[\frac{1+\sin A+i\cos A}{1+\sin A-i\cos A}\right]^n=\cos n\left(\frac{\pi}{2}-A\right)+i\sin n\left(\frac{\pi}{2}-A\right)$

Solution:

Let sin A + icos A = z

$$z=\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)$$

$$\frac{1}{z}=\frac{1}{\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)}=\cos\left(\frac{\pi}{2}-A\right)-i\sin\left(\frac{\pi}{2}-A\right)$$

$$=\text{Sin}A-i\text{Cos}A$$

$$\left[\frac{1+\sin A+i\cos A}{1+\sin A-i\cos A}\right]^n=\left[\frac{1+z}{1+\frac{1}{z}}\right]^n$$

$$=\left[\frac{1+z}{\frac{z+1}{z}}\right]^n$$

$$=\left[\frac{z(1+z)}{z+1}\right]^n=z^n$$

$$=\left[\cos\left(\frac{\pi}{2}-A\right)+i\sin\left(\frac{\pi}{2}-A\right)\right]^n$$

$$=\cos n\left(\frac{\pi}{2}-A\right)+i\sin n\left(\frac{\pi}{2}-A\right)$$

6) If $2\cos\theta=x+\frac{1}{x}$ and $2\cos\phi=y+\frac{1}{y}$ prove that one of the values of $x^m y^n+\frac{1}{x^m y^n}$ is $2\cos(m\theta+n\phi)$

Solution:

$$2\cos\theta=x+\frac{1}{x}$$

$$2x\cos\theta=x^2+1$$

$$x^2 - 2x\cos\theta = -1$$

$$x^2 - 2x\cos\theta + \cos^2\theta = -1 + \cos^2\theta$$

$$= -\left(1 - \cos^2\theta\right)$$

$$\left(x - \cos\theta\right)^2 = -\sin^2\theta = i^2\sin^2\theta$$

$$x - \cos\theta = i\sin\theta$$

$$x = \cos\theta + i\sin\theta$$

$$x^m = \left(\cos\theta + i\sin\theta\right)^m = \cos m\theta + i\sin m\theta$$

Similarly $y = \cos\phi + i\sin\phi$

$$y^n = \left(\cos\phi + i\sin\phi\right)^n$$

$$= \cos n\phi + i\sin n\phi$$

$$\therefore x^m y^n = \left(\cos m\theta + i\sin m\theta\right)\left(\cos n\phi + i\sin n\phi\right)$$

$$= \left(\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)\right)$$

$$\frac{1}{x^m y^n} = \frac{1}{\left(\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)\right)}$$

$$= \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$

$$x^m y^n + \frac{1}{x^m y^n} = \left(\cos(m\theta + n\phi) + i\sin(m\theta + n\phi)\right) + \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$$

$$= 2\cos\left(m\theta + n\phi\right)$$

2.3. ROOTS OF A COMPLEX NUMBER

DEFINITION:

A number ω is called a nth root of a complex number Z

if $\omega^n = z$ and we write $\omega = \left(z\right)^{\frac{1}{n}}$

Working rule to find the nth roots of a complex number:

Step1: write the given number in polar form.

Step2: Add $2k\pi$ to the argument.

Step3: apply De-moivre's theorem (bring the power to inside)

Step4: Put k=0,1……. up to (n-1)

Illustration:

Let $z = r(\cos\theta + i\sin\theta)$

$= r[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]$, k is an integer

$$\therefore z^{\frac{1}{n}} = [r\{\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)\}]^{\frac{1}{n}}$$

$$= (r)^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right)+i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

Where k=0,1,2,3………(n-1)

Only these values of k will give n different values of $z^{\frac{1}{n}}$ provided $z \neq 0$

Note:

(1) The number of nth roots of a non-zero complex number is n.

(2) The moduli of these roots is the same non negative real number.

(3) The argument of these n roots is equally spaced. That is if the principal valve of argument of z is θ, i.e., $-\pi<\theta\leq\pi$ then the argument of other roots of z are obtained by adding respectively $\frac{2\pi}{n}, \frac{4\pi}{n}$......to $\frac{\theta}{n}$

(4) If k be given integral values greater than or equal to n, these n values are repeated and no fresh root is obtained.

The nth roots of unity:

1 = (cos0 + isin0) = $\cos 2k\pi + i\sin 2k\pi$

nth roots of unity $= (1)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$$= \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) \text{ where } k = 0,1,2.......(n-1)$$

$\therefore$ The nth roots of unity are cos0+isin0,

$$\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n}+i\sin\frac{4\pi}{n}, \cos\frac{6\pi}{n}+i\sin\frac{6\pi}{n}, \cos(n-1)\frac{2\pi}{n}+i\sin(n-1)\frac{2\pi}{n}$$

Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e\frac{i2\pi}{n}$

Then the nth roots of unity are

$$e^0, e^{\frac{i2\pi}{n}}, e^{\frac{i4\pi}{n}}, e^{\frac{i6\pi}{n}} ....e^{\frac{i2(n-1)\pi}{n}} \quad \text{become } 1, \omega, \omega^2.......\omega^{n-1}$$

It is clear that the roots are in geometric progression.

Results:

(1) $\omega^n = 1$

$$\omega^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos 2\pi + i\sin 2\pi = 1$$

(2) Sum of the roots is zero

i.e., $1 + \omega + \omega^2 + \omega^3 + \ldots\ldots + \omega^{n-1} = 0$

$\therefore$ LHS $= 1 + \omega + \omega^2 + \ldots\ldots \omega^{n-1}$ is a G.P with n terms.

$$= \frac{1-\omega^n}{1-\omega} = \frac{1-1}{1-\omega} = 0 = \text{RHS} \therefore \left[1 + r + r^2 ..... + r^{n-1} = \frac{1-r^n}{1-r}\right]$$

3) The roots are in G.P with common ratio ω

4) The argument are in A.P with common difference $\frac{2\pi}{n}$

5) Product of the roots = $(-1)^{n+1}$

Cube roots of unity: $(1)^{\frac{1}{3}}$

Let $x = (1)^{\frac{1}{3}}$

$$\therefore x = (\cos 0 + i\sin 0)^{\frac{1}{3}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{3}} \text{ where } k \text{ is an integer}$$

$$x = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \qquad \text{where } k = 0,1,2$$

The three roots are

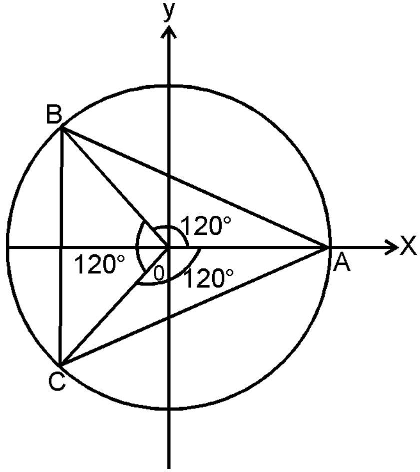
$$\cos 0 + i\sin 0,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

i.e., $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore$ The roots are $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Result:

The modulus of each root of $(1)^{\frac{1}{3}}$ is 1.



$\therefore$ All these roots lie on the circumference of the unit circle. Let A,B and C be points represented by the three roots $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ in ordered pair form. The angles between OA and OB, OB and OC and OA are each $\frac{2\pi}{3}$ radians or 120°. Hence when these points are joined by straight line they will form the vertices of an equilateral triangle.

If we denote the second root $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ by ω then the other root $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \omega^2$

Hence we observe that the cube roots of unity namely 1, ω, ω^2 are in G.P.

Note:

1. Even if $\frac{-1+i\sqrt{3}}{2}$ is taken as ω it can be proved that $\frac{-1-i\sqrt{3}}{2} = \omega^2$
2. $1+\omega+\omega^2=0$ i.e., the sum of the cube roots of unity is zero.
3. Since ω is a root of the equation $x^3=1$, we see that $\omega^3=1$.

Fourth roots of unity:

Let x be a fourth root of unity. Then $x = (1)^{\frac{1}{4}}$

$$\therefore x^4 = 1$$

$$= (\cos 2k\pi + i\sin 2k\pi) \text{ where } k \text{ is an integer}$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{4}}$$

$$x = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}$$

$$x = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2} \quad \text{where } k = 0,1,2,3$$

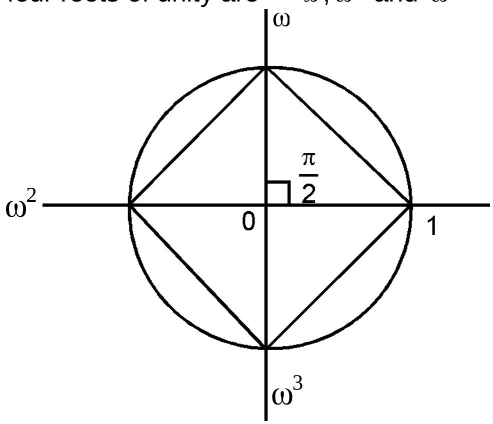
The four roots are

$$\cos 0 + i\sin 0, \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \cos\pi + i\sin\pi, \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

i.e., 1,i,-1,-i.

Let as denote $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ by ω.

Then the four roots of unity are 1 ω, ω^2 and ω^3



The fourth roots of unity form the vertices square all lying on the unit circle. We observe that the sum of the fourth roots of unity is zero. i.e., $1+\omega+\omega^2+\omega^3=0$ and $\omega^4=1$

Note: The values of ω used in cube roots of unity and in fourth roots of unity are different.

Sixth roots of unity:

Let x be a sixth root of unity. Then $x = (1)^{\frac{1}{6}}$

$$\therefore 1 = \cos 0 + i\sin 0$$

$$(1)^{\frac{1}{6}} = (\cos(0+2k\pi) + i\sin(0+2k\pi))^{\frac{1}{6}} = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{6}}$$

where k is an integer

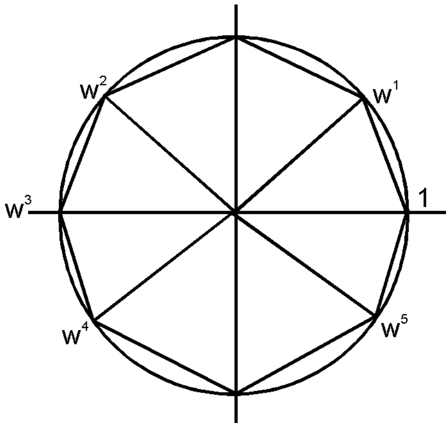
By De-moivre's theorem

$$x = (1)^{\frac{1}{6}} = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6} \text{ where } k = 0,1,2,3,4,5$$

The six roots are

$$\cos 0 + i\sin 0, \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

$$\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}, \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$



Then the six, sixth roots of unity are 1, ω, ω^2, ω^3, ω^4, ω^5.

From the figure it can be noted that the six roots of unity form the vertices of hexagon all lying on the unit circle.

2.3 WORKED EXAMPLES

PART - A

1. If ω is the cube root of unity what is the value of $1+\omega+\omega^2$

Solution:

$1+\omega+\omega^2=0$

2. If ω is the cube root of unity what is the value of ω^3

Solution:

$\omega^3=1$

3. If ω is the cube root of unity what is the value of $\omega(\omega+1)$

Solution:

$\omega(\omega+1) = \omega^2+\omega = -1$

4. If ω is the fourth root of unity what is the value of $\omega+\omega^2+\omega^3$

Solution:

$1+\omega+\omega^2+\omega^3=0$

$\omega+\omega^2+\omega^3=-1$

5. If ω is the nth root of unity what is the value of $1+\omega+\omega^2+...+\omega^{n-1}$

Solution:

$1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-1}=0$

6. Solve $x^2+1=0$

Solution:

$x^2+1=0$

$X^2 = -1$

$$x = (-1)^{\frac{1}{2}} = (\cos\pi + i\sin\pi)^{\frac{1}{2}}$$

$$= [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{\frac{1}{2}}$$

$$= \cos\left(\frac{2k+1}{2}\right)\pi + i\sin\left(\frac{2k+1}{2}\right)\pi \text{ where } k = 0,1.$$

7. Find the value of $(i)^{1/3}$

Solution:

$$\text{Let } x = (i)^{\frac{1}{3}}$$

$$x = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{\frac{1}{3}}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$= \cos\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) + i\sin\frac{1}{3}\left(2k\pi + \frac{\pi}{2}\right) \text{ where } k = 0,1,2.$$

8. Find the value of $(-1)^{1/4}$

Solution:

$$\text{Let } x = (-1)^{1/4}$$

$$x = (\cos\pi + i\sin\pi)^{1/4}$$

$$= \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/4}$$

$$= \cos\frac{1}{4}\left(2k\pi + \frac{\pi}{4}\right) + i\sin\frac{1}{4}\left(2k\pi + \frac{\pi}{4}\right)$$

where $k = 0,1,2,3...$

PART - B

1. Solve $x^3 + 1 = 0$

Solution:

$$x^3 + 1 = 0 \Rightarrow x^3 = -1$$

$$x = (-1)^{\frac{1}{3}}$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{3}}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{3}}$$

$$= \cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi \text{ where } k = 0,1,2$$

For $k = 0,\ x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k = 1,\ \ x = \cos\pi + i\sin\pi = -1$

$k = 2,\ \ x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore$ The solutions are $\frac{1}{2} + \frac{i\sqrt{3}}{2}, -1, \frac{1}{2} - \frac{i\sqrt{3}}{2}$

2) Solve: $x^5 - 1 = 0$

Solution:

$$x^5 + 1 = 0$$

$$\Rightarrow x = (1)^{1/5}$$

$$x = (\cos 0 + i\sin 0)^{\frac{1}{5}}$$

$$= [\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)]^{1/5}$$

$$= \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} \quad \text{where } k = 0,1,2,3,4$$

For $k = 0,\ \ x = \cos 0 + i\sin 0 = 1$

$k = 1,\ \ x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k = 2,\ \ x = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$k = 3,\ \ x = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$k = 4,\ \ x = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

3) Solve: $x^5 + 1 = 0$

Solution:

$$x^5 + 1 = 0$$

$$x^5 + 1 = 0 \Rightarrow x^5 = -1$$

$$x = (-1)^{1/5}$$

$$= (\cos\pi + i\sin\pi)^{1/5}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{5}}$$

$$= \cos\left(\frac{2k+1}{5}\right)\pi + i\sin\left(\frac{2k+1}{5}\right)\pi \quad \text{where } k = 0,1,2,3,4$$

For $k = 0,\ x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$k = 1,\ x = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$k = 2,\ x = \cos\pi + i\sin\pi$

$k = 3,\ x = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$k = 4,\ x = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

4) Find the value $(8i)^{1/3}$

Solution:

Let $x = (8i)^{1/3}$

$$x = 8^{1/3}(i)^{1/3}$$

$$x = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{1/3}$$

$$= 2\left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/3}$$

$$= 2\left[\cos\left(\frac{4k\pi+\pi}{2}\right)+i\sin\left(\frac{4k\pi+\pi}{2}\right)\right]^{1/3}$$

$$= 2\left[\cos\left(\frac{4k\pi+\pi}{6}\right)\pi+i\sin\left(\frac{4k\pi+\pi}{6}\right)\pi\right] \text{ where } k = 0,1,2,$$

For $k = 0, \quad x = 2\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$

$k = 1 \quad x = 2\left[\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right]$

$k = 2 \quad x = 2\left[\cos\frac{9\pi}{6}+i\sin\frac{9\pi}{6}\right]$

5) Solve $x^7+1=0$

Solution:

$x^7+1=0$

$x^7+1=0 \Rightarrow x^7 = -1$

$$x = (-1)^{\frac{1}{7}} = (\cos\pi+i\sin\pi)^{\frac{1}{7}}$$

$$= [\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{\frac{1}{7}}$$

$$= \cos\left(\frac{2k+1}{7}\right)\pi+i\sin\left(\frac{2k+1}{7}\right)\pi \quad \text{where } k = 0,1,2,3,4,5,6$$

For $k = 0, \quad x = \cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$

$k = 1, \quad x = \cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}$

$k = 2, \quad x = \frac{5\pi}{7}+i\sin\frac{5\pi}{7}$

$k = 3, \quad x = \cos\pi+i\sin\pi$

$k = 4, \quad x = \cos\frac{9\pi}{7}+i\sin\frac{9\pi}{7}$

$k = 5, \quad x = \cos\frac{11\pi}{7}+i\sin\frac{11\pi}{7}$

$k = 6, \quad x = \cos\frac{13\pi}{7}+i\sin\frac{13\pi}{7}$

6) Find the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$. Prove that the product of the four values is 1

Solution:

Let $\frac{1}{2}+i\frac{\sqrt{3}}{2}=a+ib=r(\cos\theta+i\sin\theta)$

$$a=\frac{1}{2},\ \ b=\frac{\sqrt{3}}{2}$$

$$r=\sqrt{a^2+b^2}=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$$

$$\cos\theta=\frac{a}{r}=\frac{1}{2},\ \ \sin\theta=\frac{b}{r}=\frac{\sqrt{3}}{2}$$

$$\therefore \theta=60^0=\frac{\pi}{3}$$

$$\therefore \frac{1}{2}+i\frac{\sqrt{3}}{2}=1\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}=\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\right]^{\frac{1}{4}}$$

$$=(\cos\pi+i\sin\pi)^{\frac{1}{4}}=[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{\frac{1}{4}}$$

$$=\cos\left(\frac{2k+1}{4}\right)\pi+i\sin\left(\frac{2k+1}{4}\right)\pi \quad \text{where} \quad k=0,1,2,3,$$

For $k=0$, $R_1=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$

$k=1$, $R_2=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}$

$k=2$, $R_3=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}$

$k=3$, $R_4=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}$

The product is

$$R_1R_2R_3R_4 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$$

$$= \mathrm{Cos}\frac{16\pi}{4} + i\sin\frac{16\pi}{4}$$

$$= \cos 4\pi + i\sin 4\pi = 1$$

7) Solve: $x^5 + x^3 + x^2 + x = 0$

Solution:

$$x^5 + x^3 + x^2 + x = 0$$

$$x^3(x^2 + 1) + (x^2 + 1) = 0$$

$$(x^2 + 1)(x^3 + 1) = 0$$

$$\therefore x^2 + 1 = 0 \quad \text{and} \quad x^3 + 1 = 0$$

Case 1

$$x^2 + 1 = 0 \Rightarrow x^2 = -1$$

$$x = (-1)^{\frac{1}{2}} = (\cos\pi + i\sin\pi)^{\frac{1}{2}}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{2}}$$

$$= \cos\left(\frac{2k+1}{2}\right)\pi + i\sin\left(\frac{2k+1}{2}\right)\pi \quad \text{where} \quad k = 0,1$$

For $k = 0, \quad x = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$k = 1, \quad x = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

Case: 2

$$x^3+1=0$$

$$x^3=-1 \Rightarrow x=(-1)^{\frac{1}{3}}$$

$$x=(\cos\pi+i\sin\pi)^{\frac{1}{3}}$$

$$=[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{\frac{1}{3}}$$

$$=\cos\left(\frac{2k+1}{3}\right)\pi+i\sin\left(\frac{2k+1}{3}\right)\pi \quad \text{where} \quad k=0,1,2$$

For $k=0,\ x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

$k=1 \quad x=\cos\pi+i\sin\pi$

$k=2 \quad x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

8) Solve $x^8-x^5+x^3-1=0$

Solution:

$$x^8-x^5+x^3-1=0$$

$$x^5(x^3-1)+(x^3-1)=0$$

$$(x^3-1)(x^5+1)=0$$

$$x^3-1=0 \quad \text{and} \quad x^5+1=0$$

Case: 1

1 $x^3-1=0$

$$x^3=1\Rightarrow x=(1)^{\frac{1}{3}}$$

$$x=(\cos 0+i\sin 0)^{\frac{1}{3}}$$

$$=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{3}}$$

$$x=\frac{\cos 2k\pi}{3}+\frac{i\sin 2k\pi}{3} \quad \text{where} \quad k=0,1,2$$

For $k=0,\quad x=\cos 0+i\sin 0=1$

$k=1 \quad x=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$k=2 \quad x=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

Case: 2

$$x^5 + 1 = 0$$

$$x^5 = -1$$

$$x = (-1)^{\frac{1}{5}} = (\cos\pi + i\sin\pi)^{1/5}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/5}$$

$$= \cos\left(\frac{2k+1}{5}\right)\pi + i\sin\left(\frac{2k+1}{5}\right)\pi \quad \text{where } k = 0,1,2,3,4$$

For $k = 0, \quad x = \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}$

$k = 1, \quad x = \cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5}$

$k = 2, \quad x = \cos\pi + i\sin\pi$

$k = 3, \quad x = \cos\dfrac{7\pi}{5} + i\sin\dfrac{7\pi}{5}$

$k = 4, \quad x = \cos\dfrac{9\pi}{5} + i\sin\dfrac{9\pi}{5}$

EXERCISE
PART - A

1. State Demoivre's theorem.
2. Write the value of ($(\cos\theta + i\sin\theta)^{-n}$
3. Simplify:

   (a) $(\cos\theta + i\sin\theta)^3$ (b) $(\cos\theta - i\sin\theta)^8$

   (c) $(\cos4\theta - i\sin4\theta)^{-5}$ (d) $(\cos\theta - i\sin\theta)^{-7}$

   (e) $(\cos\theta + i\sin\theta)^{\frac{3}{5}}$ (f) $(\cos\theta + i\sin\theta)^{\frac{-5}{3}}$
4. Simplify the following:

   (i) $(\cos\theta + i\sin\theta)^3 \times (\cos\theta + i\sin\theta)^{-4}$

   (ii) $(\cos x + i\sin x)^5 \times (\cos x + i\sin x)^6$

5. Find the value of the following:

(a) $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 4\theta - i\sin 4\theta}$ (b) $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta - i\sin 3\theta}$

(c) $\dfrac{(\cos\theta + i\sin\theta)^8}{(\cos 2\theta + i\sin 2\theta)^7}$ (d) $\dfrac{\cos 5\theta - i\sin 5\theta}{\cos 6\theta + i\sin 6\theta}$

(e) $\dfrac{\cos 6\theta + i\sin 6\theta}{(\cos\theta - i\sin\theta)^3}$

6. If $z = \cos 30^0 + i\sin 30^0$ find z^3

7. If $z = \cos 100^0 + i\sin 100^0$ find $z^{\frac{1}{5}}$

8. If $z = \cos\theta + i\sin\theta$ find z^n

9. If $x = \cos\theta + i\sin\theta$ find

(i) $x + \dfrac{1}{x}$ (ii) $x^3 + \dfrac{1}{x^3}$ (iii) $x^n + \dfrac{1}{x^n}$

10. Solve $x^2 - 1 = 0$

11. If ω is the cube roots of unity, what is the value of $1+\omega$.

12. If ω is the fourth roots of unity, what is the value of $1+\omega+\omega^2$

13. If ω is the sixth roots of unity, what is the value of $1+\omega^2+\omega^3+\omega^4$

PART - B

1. Simplify the following using Demoivre's theorem

(a) $\dfrac{(\cos 2\theta - i\sin 2\theta)^4 (\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 5\theta - i\sin 5\theta)^{-3}}$

(b) $\dfrac{(\cos 2\theta - i\sin 2\theta)^3 (\cos 3\theta + i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 (\cos 5\theta - i\sin 5\theta)^{-3}}$

(c) $\dfrac{(\cos\theta - i\sin\theta)^3 (\cos 3\theta + i\sin 3\theta)^5}{(\cos 2\theta - i\sin 2\theta)^5 (\cos 5\theta + i\sin 5\theta)^7}$ where $\theta = \dfrac{2\pi}{13}$

(d) $\dfrac{(\cos 2x - i\sin 2x)^{-8}(\cos 3x + i\sin 3x)^{-4}}{(\cos 4x + i\sin 4x)^{2}(\cos 5x - i\sin 5x)^{3}}$

(e) $\dfrac{(\cos 4\theta + i\sin 4\theta)^{2}(\cos 3\theta - i\sin 3\theta)^{2}}{(\cos\theta - i\sin\theta)^{7}(\cos 2\theta + i\sin 2\theta)^{-5}}$ when $\theta = \dfrac{\pi}{16}$

(f) $\dfrac{(\cos 5\theta - i\sin 5\theta)\ (\cos 2\theta - i\sin 2\theta)^{-3}}{(\cos\theta + i\sin\theta)^{5}(\cos 3\theta + i\sin 3\theta)^{-5}}$ when $\theta = \dfrac{2\pi}{11}$

2. Show that $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^{4} = 1$

3. Show that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\sin\theta - i\cos\theta}\right]^{3} = \cos 3\theta + i\sin 3\theta$

4. Show that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\sin\theta - i\cos\theta}\right]^{5} = \cos 5\theta + i\sin 5\theta$

5. Express $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^{5}$ in $x + iy$ form

6. Show that $\left(1+i\sqrt{3}\right)^{4} = 16\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$

7. If $a = \cos x + i\sin x$ and $b = \cos y + i\sin y$ prove that $\sqrt{ab} + \dfrac{1}{\sqrt{ab}} = 2\cos\left(\dfrac{x+y}{2}\right), \sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}} = 2\cos\left(\dfrac{x-y}{2}\right)$.

8. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ find the value of $\dfrac{a^2}{b^2} - \dfrac{b^2}{a^2}$

9. If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$, and $c = \cos z + i\sin z$ find the value of (i) $\dfrac{ab}{c} + \dfrac{c}{ab}$ (ii) $\dfrac{ab}{c} - \dfrac{c}{ab}$.

10. If $x = \cos 3\alpha + i \sin 3\alpha$, $b = \cos y + i \sin y$, find the value of

(i) $\sqrt[3]{xy} + \dfrac{1}{\sqrt[3]{xy}}$ (ii) $\sqrt[3]{xy} - \dfrac{1}{\sqrt[3]{xy}}$

11. When 'n' is an integer, prove that

$$\left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n = 2^{n+1}\cos\frac{n\pi}{3}$$

12. Evaluate: $\left(1+i\sqrt{3}\right)^8 + \left(1-i\sqrt{3}\right)^8$

13. When 'n' is a positive integer, prove that

$$(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$$

14. Evaluate: $(1+i)^4 + (1-i)^4$

15. If $(a_1 + ib_1)(a_2 + ib_2)\ldots\ldots\ldots\ldots(a_n + ib_n) = A + iB$ Prove that

(i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots\ldots\ldots(a_n^2 + b_n^2) = A^2 + B^2$

(ii) $\tan^{-1}\left[\dfrac{b_1}{a_1}\right] + \tan^{-1}\left[\dfrac{b_2}{a_2}\right] + \ldots\ldots\ldots\ldots + \tan^{-1}\left[\dfrac{b_n}{a_n}\right] = n\pi + \tan^{-1}\left[\dfrac{B}{A}\right]$

16. If $\sqrt{x - iy} = A - iB$ where $A > 0$, prove that $A^2 - B^2 = x, 2AB = y$

17. Find the value of $(i)^{\frac{2}{3}}$

18. Find the cube roots of 8

19. Find the value of $(1+i)^{\frac{1}{4}}$

20. Find the value of $(1+i\sqrt{3})^{\frac{1}{4}}$

21. Solve $x^3 - 1 = 0$ (or) Find the cube roots of unity

22. Solve $x^7 - 1 = 0$

23. Solve $x^4 + 81 = 0$

24. Solve $x^4 + 4 = 0$

25. Solve $x^7 + x^4 + x^3 + 1 = 0$

26. If $1, \omega, \omega^2$ are the cube roots of unity prove that

(i) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$

(ii) $(2 - \omega)(2 - \omega^2)(2 - \omega^9)(2 - \omega^{11}) = 49$

(iii) Prove that $\frac{1}{1+2\omega} - \frac{1}{1+\omega} + \frac{1}{2+\omega} = 0$

ANSWERS
PART - A

(2) $\cos n\theta - i\sin n\theta$

(3) (a). $\cos 3\theta + i\sin 3\theta$

(b) $\cos 8\theta - i\sin 8\theta$

(c). $\cos 20\theta + i\sin 20\theta$

(d). $\cos 7\theta - i\sin 7\theta$

(e). $\cos\frac{3}{5}\theta - i\sin\frac{3}{5}\theta$

(f). $\cos\frac{5}{3}\beta - i\sin\frac{5}{3}\beta$

(4) (i). $\cos\theta - i\sin\theta$

(ii) $\cos 11x + i\sin 11x$

(5) (a) $\cos 7\theta + i\sin 7\theta$

(b) $\cos 5\theta + i\sin 5\theta$

(c) $\cos 22\theta + i\sin 22\theta$

(d) $\cos 11\theta - i\sin 11\theta$

(e) $\cos 9\theta + i\sin 9\theta$

(7) $\cos 20^0 + i\sin 20^0$

(8) $\cos n\theta + i\sin n\theta$

(9) (i) $2 \cos \theta$

(ii) $2i\sin 3\theta$

(iii) $2 \cos n\theta$

(10) $\cos k\pi + i\sin k\pi,\ k = 0,1$

(11) $-\omega^2$

12.) $-\omega^3$

(13) $-\omega^4 - \omega^5$

ANSWERS
PART - B

(1) (a) $\cos 49\theta - i\sin 49\theta$ (b) $\cos 15\theta + i\sin 15\theta$ **(c)** 1(d) $\cos 11x + i\sin 11x$ (e) $\cos\frac{19}{16}\pi + i\sin\frac{19}{16}\pi$ **(f)** 1

5.) $\sin 10\theta - i\cos 10\theta$ 8.) $2i\sin(2\alpha - 2\beta)$

(9.).**(i)** $2\cos(x+y-z)$ **(ii)** $2i\sin(x+y-z)$

10.) **(i)** $2\cos(\alpha+\beta)$ **(ii)** $2i\sin(\alpha+\beta)$

12.) $2^9\cos\frac{8\pi}{3}$ 14.) -8

17.) $\cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi\ k = 0,1,2.$

18.) $2\left[\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right]\ k = 0,1,2$

19.) $2^{\frac{1}{8}}\left[\cos\left(\frac{8k+1}{16}\right)\pi + i\sin\left(\frac{8k+1}{16}\right)\pi\right]\ k = 0,1,2,3.$

20.) $2^{\frac{1}{4}}\left[\cos\left(\frac{6k+1}{12}\right)\pi + i\sin\left(\frac{6k+1}{12}\right)\pi\right]\ k = 0,1,2,3.$

21.) $\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\quad k = 0,1,2.$

22.) $\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}\quad k = 0,1,2,3,4,5,6.$

23.) $3\left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right] k = 0,1,2,3$

24.) $2^{\frac{1}{2}}\left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)\right], k = 0,1,2,3$

25.) $\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right), k = 0,1,2,3$

$\cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right), k = 0,1,2$

UNIT- III

PROBABILITY DISTRIBUTION - II

POISSON DISTRIBUTION

3.1 Definition:- $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!} \quad x=0,1,2,.......$

(Statement only). Expression for mean and Variance. Simple problems.

NORMAL DISTRIBUTION

3.2 Definition of normal and standard normal distribution. (Statement only). Constants of normal distribution (results only) – Properties of normal distribution – Simple problems using the table standard normal distribution.

CURVE FITTING

3.3 Fitting of a straight line using least square method (result only) – simple problems.

3.1 POISSON DISTRIBUTION

Introduction: Poisson distribution was discovered by the French Mathematician and Physicist Simeon Denis Poisson (1781-1840) in the year 1837. Poisson distribution is the discrete distribution.

Poisson distribution is a limiting case of the Binomial distribution under the following conditions:

i. n, the number of trials is indefinitely large; i.e., $n \to \infty$

ii. p, the probability of success for each trial is sufficiently small;

i.e., $p \to 0$

iii. np, $=\lambda$ (say), is finite.

Definition: The probability function is the Poisson distribution if

$$P(X=x)=\begin{cases}\dfrac{e^{-\lambda}\lambda^{x}}{x!} & x=0,1,2,....... \\ 0 & \text{otherwise}\end{cases}$$

Note:

i. Poisson distribution is a discrete probability distribution, since the variable X can take only integral values 0,1,2…..

ii. λ is known as the parameter of the distribution.

Constants of Poisson distribution:

Mean $=\lambda$

Variance $=\lambda$

Standard deviation $=\sqrt{\lambda}$

Examples of Poisson Distribution:

1. Number of printing mistakes at each page of the book
2. Number of defective blades in a packet of 150.
3. Number of babies born blind per year in the city.
4. Number of air accidents in some unit of time.
5. Number of suicides reported in a particular city in 1 hour.
6. Number of deaths due to snake bite in some unit of time.

We use the notation $X \sim p(\lambda)$ to denote that X is a Poisson varitate with parameter. λ

3.1 WORKED EXAMPLES
PART - A

1.) The Probability of a Poisson variable taking the values 3 and 4 are equal. Calculate the value of the parameter λ and the standard deviation.

Solution:

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!} \quad x=0,1,2,.......$$

Given $P(X=3)=P(X=4)$

$$\frac{e^{-\lambda}\lambda^{3}}{3!}=\frac{e^{-\lambda}\lambda^{4}}{4!}$$

$$\frac{1}{6}=\frac{\lambda}{24} \quad \lambda=4$$

$$SD=\sqrt{\lambda}=\sqrt{4}=2$$

2.) For a Poisson distribution with n=1000, and λ =1 find p.

Solution:

$np = \lambda$

$1000\,p = 1$

$$P = \frac{1}{1000} = 0.001.$$

3.) Criticise the following statement

"The mean of a Poisson distribution is 5 while the standard deviation is 4"

Solution:

Let λ be the parameter of the Poisson distribution

mean = λ and S.D.= $\sqrt{\lambda}$

$$\therefore \text{ S.D.} = \sqrt{\text{mean}}$$

$$4 = \sqrt{5} \text{ which is not possible.}$$

4.) Find the Probability that no defective fuse will be found in a box of 200 fuses if experience show that 2% such fuses are defective.

Solution:

The Poisson distribution is $P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$, x =,01,2,3,.........

Let X denote the defective fuse.

$p = 2\% = \frac{2}{100}$ n=200

mean λ=n p= $\frac{2}{100} \times 200 = 4$

P(no defective fuse) = P(X=0)

$$= \frac{e^{-\lambda}\lambda^{0}}{o!} = e^{-4}$$

PART- B

1.) Let X is a Poisson variate such that P(X=1) = 0.2 and P(X=2) 0.15, find P(X=0).

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!} \quad x=0,1,2,.......$$

Given P(X=1) = 0.2 P(X=2) = 0.15

$$\frac{e^{-\lambda}\lambda^{1}}{1!}=0.2 \qquad \frac{e^{-\lambda}\lambda^{2}}{2!}=0.15$$

$$e^{-\lambda}\lambda=0.2 \qquad \ldots 1$$

$$e^{-\lambda}\lambda^{2}=0.3 \qquad \ldots 2$$

$$\frac{(2)}{(1)}=-\frac{e^{-\lambda}\lambda^{2}}{e^{-\lambda}\lambda}=\frac{0.3}{0.2}=\frac{3}{2}$$

λ=1.5

$$\therefore P(X=0)=\frac{e^{-\lambda}\lambda^{0}}{0!}=e^{-\lambda}=e^{-1.5}=0.2231$$

2.) The Probability that a man aged 50 years will die within a years is 0.01125. What is the probability that of 12 such men at least 11 will reach their fifty first birthday?

Solution:

Since the probability of death is very small, we use poisson distribution. Here p=0.01125 and n=12.

Mean λ= np

= 0.01125×12

= 0.135

P (atleast 11 persons will survive)=P($X \le 11$)

=P (at most one person dies)

=P($X \le 1$)

=P(X=0)+P(X=1)

$$=\frac{e^{-\lambda}\lambda^{0}}{0!}+\frac{e^{-\lambda}\lambda^{1}}{1!}$$

$$=e^{-\lambda}(1+\lambda)=e^{-.135}(1+0.135)=0.9916.$$

3.) The number of accidents in a year involving taxi drivers in a city follow a Poisson distribution with mean equal to 3. Out of 1000 taxi drivers find approximately the number of drivers with (i) no accident in a year (ii) more then 3 accident in a year $\left[e^{-3} = 0.0498\right]$

Solution:

Let X denote number of accident in a year involving taxi drivers,

Given mean λ=3

(i) P(no. of accident in a year) =P (X=0)

$$= \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = e^{-3} = 0.0498$$

No. of taxi drivers with no accidents = 1000 $\times$0.0498

= 49.8=50(approx)

(ii) P(more than 3 accidents) =P(X>3)

=1- P(X$\leq$3)

$$= 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3)\}$$

$$= 1 - \left\{\frac{e^{-\lambda}\lambda^0}{o!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^3}{3!} + \right\}$$

$$= 1 - e^{-3}\left\{1 + 3 + \frac{9}{2} + \frac{27}{6}\right\}$$

$$= 1 - e^{-3}(13) = 1 - 0.0498x13 = 0.3526$$

No. of taxi drivers with more than 3 accidents=1000 $\times$0.3526

=352.6

=353drivers (approx)

4.) 20% of the bolts produced in a factory are found to be defective. Find the probability that in a sample of 10 bolts chosen at random exactly 2 will be defective using. (i) Bionomial distribution (ii) Poission distribution.

Let X denote the number of bolts produced to be defective

$$p = 20\% = \frac{20}{100} = \frac{1}{5}$$

$$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5} \qquad n = 10$$

(i) Using Binomial Distribution, $P(X=2) = 10C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8$

$$= 45.\frac{4^8}{5^{10}} = 0.3020$$

(iii) Using Poisson Distribution, $\lambda = np = 10 \times \frac{1}{5} = 2$

$$P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-2}.2^2}{2} = 0.1353 \times 2 = 0.2706$$

3.2 NORMAL DISTRIBUTION

Introduction: In this unit we deal with the most important continuous distribution, known as normal distribution.

The normal distribution was first discovered by the English mathematician De – Moivre (1667-1754) in 1733 as a limiting case of the binomial distribution. The normal distribution is also known as Gaussian distribution in honour of Karl friedrich Gauss.

Definition: A Continuous random variable X is said to be normally distributed if it has the probability density function represented by the equation.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -(1) \qquad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0$$

Here μ and σ, the parameters of distribution are respectively the mean and the standard deviation of the normal distribution. The function f(x) is called the probability density function of the normal distribution and is called the normal variable. The probability distribution is sometimes briefly denoted by symbol $N(\mu, \sigma^2)$

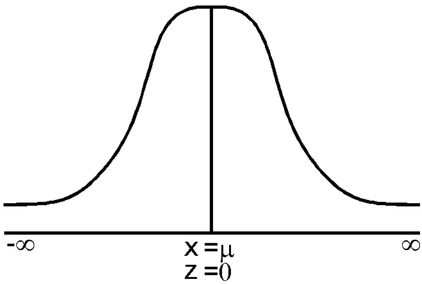
Constants of Normal Distribution:

Mean = μ

Variance = σ^2

Standard deviation = σ

The graph (shape) of the function given by (1) is called normal probability curve or simply normal curve and is shown in the following diagram.



Properties of Normal Distribution:

1.) The normal curve is perfectly symmetrical about the mean. This means that if we fold the curve along the vertical line at μ, the two halves of the curve would coincide. Further the curve is bell shaped.

2.) Mean, median and mode of the distribution coincides. Thus mean = median= mode = μ

3.) It has only one mode at x= μ Hence it is unimodal.

4.) The maximum ordinate is at x=μ. Its value is $\frac{1}{\sigma\sqrt{2\pi}}$

5.) Since the curve is symmetrical, Skewness is zero.

6.) The points of inflection of the normal curve are x= $\mu \pm \sigma$

7.) X-axis is an asymptote to the curve i.e., as the distance of the curve from the mean increases, the curve comes closer and closer to the axis and never touches it.

8.) The ordinate at the mean of the distribution divides the total area under the normal curve into two equal parts.

Standard normal distribution:

A random variable z is called a standard normal variable if its mean is zero and its standard deviation is one.

The normal distribution with mean zero and standard deviation one is known as standard Normal Distribution.

The Probability density function of the standard normal variate is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad -\infty < z < \infty$$

Where $z = \dfrac{x-\mu}{\sigma}$

The Standard Normal Distribution is usually denoted by N(0,1)

Remark: (i) Normal Distribution is a limiting form of the Binomial Distribution under the following Conditions.

a) n, the number of trials is infinitely large i.e.$n\to\infty$ and

b) neither p nor q is very small

(ii) Normal distribution can also be obtained as a limiting form of Poisson distribution with parameter $\lambda \to \infty$

Note:

The table of area (Probabilities) under the standard normal distribution is given at the end of the this unit.

3.2 WORKED EXAMPLES

PART - A

1.) Let z be a standard normal variate. Calculate following Probability

(i) $P(0 \le z \le 1.2)$ (ii) $P(-1.2 \le z \le 0)$ (iii) Area right of z = 1.3

(iv) Area left of $z = 1.5$ (v) $P(-1.2 \le z \le 2.5)$

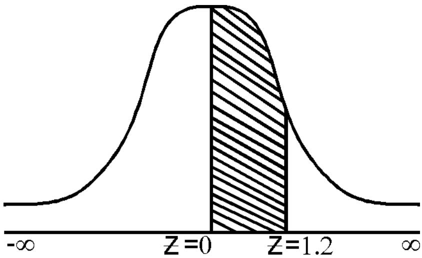
(vi) $P(-1.2 \le z \le -0.5)$ (vii) $P(1.5 \le z \le 2.5)$

Solution:

(i) $P(0 \le z \le 1.2)$

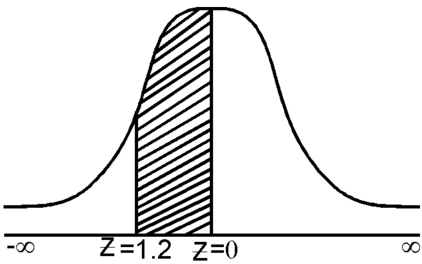
$P(0 \le z \le 1.2)$ = Area between $z = 0$ and $z = 1.2$

$= 0.3849$



(ii) $P(-1.2 \le z \le 0) = P(0 \le z \le 1.2)$ (By symmetry)

$= 0.3849$



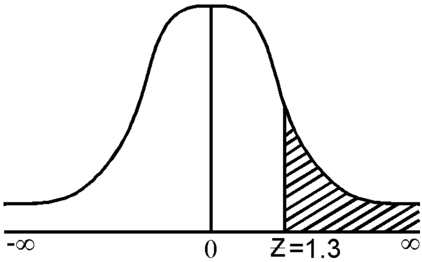
(iii) Area to the right of $z = 1.3$

$P(z > 1.3)$ = Area between $z = 0$ to $z = \infty -$ Area between $z = 0$ to $z = 1.3$

$= P(0 < z < \infty) - P(0 \le z < 1.3)$

$= 0.5 - 0.4032$

$= 0.0968$



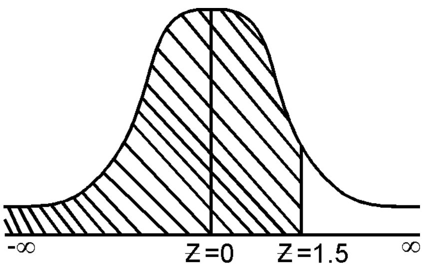
(iv) Area to the left of $z = 1.5$

$= P(z < 1.5)$

$= P(-\infty < z < 0) + P(0 \le z < 1.5)$

$= 0.5 + 0.4332$

$= 0.9332$

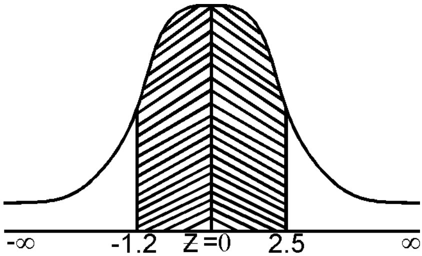


v. $P(-1.2 \le z \le 2.5)$

$P(-1.2 < z < 0) + P(0 < z < 2.5)$

$P(0 < z < 1.2) + P(0 < z < 2.5)$ (By symmetry)

$= 0.3849 + 0.4938$ $= 0.8787$



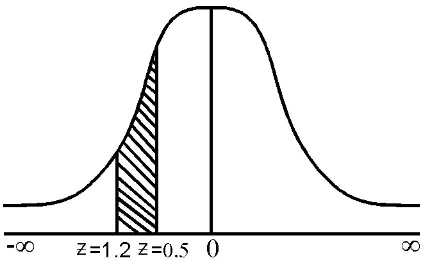
vi $P(-1.2 \le z \le -0.5)$

$= P(-1.2 < z < 0) - P(-0.5 < z < 0)$

$= P(0 < z < 1.2) - P(0 < z < 0.5)$

=0.3849 - 0.1915 (By symmetry)

=0.1934

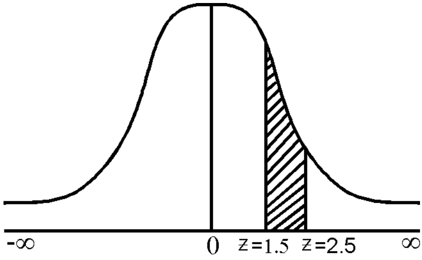


vii $P(1.5 \le z \le 2.5)$

$= P(0 \le z \le 2.5) - p(0 \le z \le 1.5)$

=0.4938-0.4332

=0.0606



2.) If z is a standard normal variate, find the value of C for the following (i) $P(0 < z < C) = 0.25$ (ii) $P(-C < z < C) = 0.40$

(iii) $P(z > C) = 0.85$

Solution:

(i) $P(0 < z < C) = 0.25$

$\Rightarrow C = 0.67$ (from the tables)

(ii) $P(-C < z < C) = 0.40$

$$P(-C < z < 0) + |P(0 < z < C)| = 0.40$$
$$P(0 < z < C) + |P(0 < z < C)| = 0.40$$

$$2P(0 < z < C) = 0.40$$
$$P(0 < z < C) = 0.20$$
$\Rightarrow C = 0.52$ (from the tables)

(iii) $P(z > C) = 0.85$

$$\Rightarrow P(0 < z < C) = 0.35$$
$$\Rightarrow C = -1.4$$

3.) In a normal distribution mean is 12 and the standard deviation is 2. Find the probability in the interval from x = 9.6 to x = 13.8

Solution:

Given mean $\mu = 12$

S.D $\sigma = 2$

To find $P(9.6 < X < 13.8)$

When x =9.6, $z = \dfrac{X-\mu}{\sigma} = \dfrac{9.6-12}{2} = \dfrac{-2.4}{2} = -1.2$

When x =13.8, $z = \dfrac{X-\mu}{\sigma} = \dfrac{13.8-12}{2} = \dfrac{1.8}{2} = 0.9$

$\therefore P(9.6 < X < 13.8) = P(-1.2 < z < 0.9)$ = 0.3849 + 0.3159 = 0.7008

PART - B

1.) If X is normally distributed with mean 6 and standard deviation 5, find (i) $P(0 \le X \le 8)$ (ii) $P(|X-6| < 10)$

Solution:

Given mean $\mu = 6$ S.D $\sigma = 5$

i. $P(0 \le X \le 8)$

When X = 0, $z = \dfrac{X-\mu}{\sigma} = \dfrac{0-6}{5} = -1.2$

When X = 8, $z = \dfrac{x-\mu}{\sigma} = \dfrac{8-6}{5} = \dfrac{2}{5} = 0.4$

$$\begin{aligned} P(0 \le X \le 8) &= P(-1.2 < z < 0.4) \\ &= P(-1.2 < z < 0) + P(0 < z < 0.4) \\ &= P(0 < z < 1.2) + P(0 < z < 0.4) \quad \text{(due to symmetry)} \\ &= 0.3849 + 0.1554 = 0.5403 \end{aligned}$$

ii. $$\begin{aligned} P(|X-6| < 10) &= P(-10 < X-6 < 10) \\ &= P(-4 < X < 16) \end{aligned}$$

When X=-4, $z = \dfrac{X-\mu}{\sigma} = \dfrac{-4-6}{5} = \dfrac{-10}{5} = -2$

When X=16, $z = \dfrac{16-6}{5} = \dfrac{10}{5} = 2$

$$\begin{aligned} P(-4 < X < 16) &= P(-2 < z < 2) \\ &= P(-2 < z < 0) + P(0 < z < 2) \\ &= P(0 < z < 2) + P(0 < z < 2) \\ &= 2(0.4772) \\ &= 0.9544. \end{aligned}$$

2.) Obtain K, μ and σ^2 of the normal distribution whose probability distribution function is given by $f(x) = Ke^{-2x^2+4x} \quad -\infty < x < \infty$

Solution:

The normal distribution is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

Consider $\quad -2x^2 + 4x = -2(x^2 - 2x) = -2(x^2 - 2X + 1 - 1)$

$$= -2[(x-1)^2 - 1]$$

$$= -2(x-1)^2 + 2$$

$$\therefore e^{-2x^2+4x} = e^{-2(x-1)^2+2}$$

$$= e^2 . e^{\frac{-1}{2}\left(\frac{x-1}{\frac{1}{2}}\right)^2}$$

Comparing with f(x), we get,

$$K e^{-2x^2 \times 4x} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$K \; e^2 e^{\frac{-1}{2}\left(\frac{x-1}{\frac{1}{2}}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

We get, $\sigma = \frac{1}{2}, \mu = 1$ and $e^2 K = \frac{1}{\sigma\sqrt{2\pi}}$

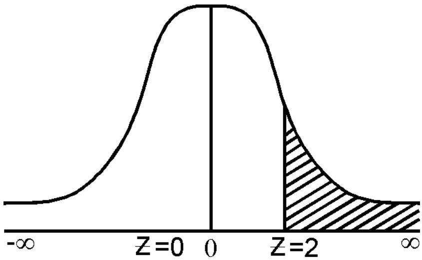
$$K = \frac{1}{\frac{1}{2}\sqrt{2\pi}} . \frac{1}{e^2}$$

$$= \frac{2e^{-2}}{\sqrt{2\pi}}$$

3.) The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12months.

Solutions:

Given, mean $\mu = 8$ and SD $\sigma = 2$



To Find $P(X > 12)$

When $X = 12$, $z = \dfrac{X - \mu}{\sigma} = \dfrac{12 - 8}{2} = \dfrac{4}{2} = 2$

$$
\begin{aligned}
P(X > 12) &= P(z > 2) \\
&= 0.5 - P(0 < z < 2) \\
&= 0.5 - 0.4772 \\
&= 0.0228
\end{aligned}
$$

$\therefore$ No, of shoes $= 5000 \times 0.0228$

=114 are in good Condition

$\therefore$ No. of shoes to be replaced after 12 months = 5000-114

= 4886 shoes

4.) Find C, μ and σ^2 of the normal distribution whose probability function is given by $f(x) = Ce^{-x^2+3x}, -\infty < x < \infty$

Solution:

The normal distribution is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

consider $-x^2 + 3x = -(x^2 - 3x)$

$$= -\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)$$

$$= -\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right)$$

$$\therefore e^{-x^2+3x} = e^{-\left(x-\frac{3}{2}\right)^2 + \frac{9}{4}}$$

$$= e^{\frac{9}{4}}.e^{-\left(x-\frac{3}{2}\right)^2} = e^{\frac{9}{4}}.e^{-\left(x-\frac{3}{2}\right)^2}$$

Comparing,

$$Ce^{-x^2+3x} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$Ce.^{\frac{9}{4}}.e^{-\left(x-\frac{3}{2}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$C.e.^{\frac{9}{4}}.e^{-\frac{1}{2}\left(\frac{x-\frac{3}{2}}{\frac{1}{\sqrt{2}}}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

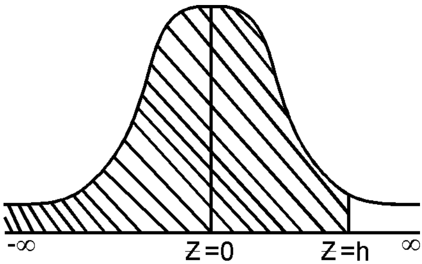
$$\therefore \mu = \frac{3}{2}, \sigma = \frac{1}{\sqrt{2}} \qquad Ce^{\frac{9}{4}} = \frac{1}{\sigma\sqrt{2\pi}}$$

$$C = \frac{1}{\frac{1}{\sqrt{2}}\sqrt{2}.\sqrt{\pi}}.e^{-9/4} = \frac{e^{-9/4}}{\sqrt{\pi}}$$

5.) If the height of 300 Students are normally distributed with mean 64.5 inches and standard deviation 3.3 inches. Find the height below which 99% of the students lie.

Solution:

Given mean $\mu = 64.5$, SD $\sigma = 3.3$ let h denote the height of students



To find $P(z \leq h) = 0.99$

$$P(0 < z < h) = 0.49$$

$$\Rightarrow h = 2.33 \qquad \text{(from the tables)}$$

$$h = \frac{X - \mu}{\sigma}$$

$$2.33 = \frac{X - 64.5}{3.3}$$

$$7.686 = X - 64.5$$

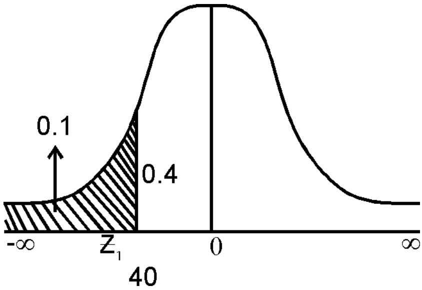
$$X = 72.189$$

$$X = 72.19$$

6.) Marks in an aptitude test given to 800 students of a schools was found to be normally distributed. 10% of the students scored below 40 marks and 10% of the students scored above 90 marks. Find the number of students scored between 40 and 90.

Solution:

Let μ be the mean & σ be the S.D



Given 10% of the students scored below 40.

$$P(z < z_1) = 0.1$$

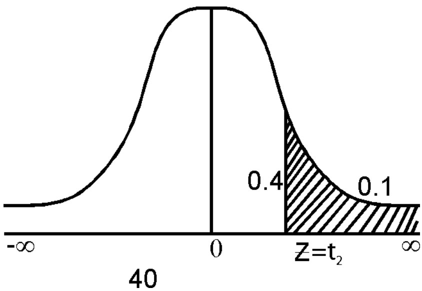
$$\Rightarrow P(0 < z < z_1) = 0.4$$

$$z_1 = -1.28 \qquad \text{(from the tables)}$$

$$z_1 = \frac{X - \mu}{\sigma}$$

$$-1.28 = \frac{40 - \mu}{\sigma} \Rightarrow 40 - \mu = -1.28\sigma \quad \text{———} \qquad \ldots 1$$

Given 10% of the students scored above 90 marks.



$$P(z > z_1) = 0.1$$

$$\Rightarrow p(0 < z < z_2) = 0.4$$

$$z_2 = 1.28$$

$$z_2 = \frac{X - \mu}{\sigma}$$

$$1.28 = \frac{90-\mu}{\sigma} \Rightarrow 90-\mu = 1.28.\sigma \quad \text{———} \qquad \ldots 2$$

Solving,

$$\begin{aligned} 90-\mu &= 1.28.\sigma \\ 40-\mu &= -1.28.\sigma \\ \hline 50 \quad &= 2.56\,\sigma \end{aligned} \qquad \Rightarrow \sigma = 19.53$$

Sub σ = 19.53 in (2)

$$90-\mu = 1.28\,(19.53)$$

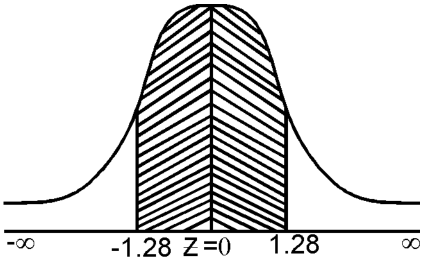
$$90-\mu = 24.998$$

$$\mu = 65$$

To Find $P(40 < X < 90)$

When $X = 40$, $\quad z = \frac{X-\mu}{\sigma} = \frac{40-65}{19.53} = 1.28$

When X=90, $z = \frac{90-65}{19.53} = 1.28$



$$P(40 < x < 90) = P(-1.28 < z < 1.28)$$

$$= 2P(0 < z < 1.28)$$

$$2(0.3997)$$

$$= 0.7994$$

$\therefore$ No. of students $= (0.7994)800$

$$= 639.52$$

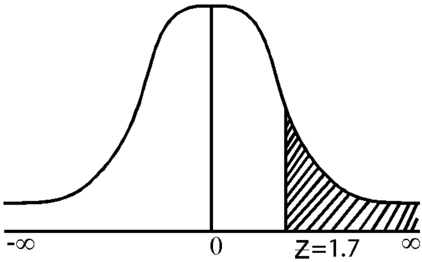
$$= 640 \text{ students}$$

7.) In a test on electric light bulbs , it was found that the life –time of a particular make was distributed normally with an average life of 2000 hours and a standard deviation of 60 hours. What proportion of bulbs can be expected to burn for more than 2100 hours.

Solution:

Given, mean $\mu = 2000$ SD $\sigma = 60$

To find $P(X > 2100)$



When $x = 2100$ $\quad z = \dfrac{x-\mu}{\sigma} = \dfrac{2100-2000}{60} = \dfrac{100}{60} = 1.7$

$$P(X > 2100) = P(z > 1.7)$$
$$= P(0 < z < \infty) - P(0 < z < 1.7)$$
$$= 0.5 - 0.4554$$
$$= 0.0446$$

$\therefore$ 4.46% of the bulbs will burn for more than 2100 hours.

3.3 CURVE FITTING

Introduction:

The graphical method and the method of group averages, are some methods of fitting curves. The graphical method is a rough method and in the method of group average, the evaluations of constants vary from one grouping to another grouping of data. So, we adopt another method of least squares which gives a unique set of values to the constants in the equation of the fitting curve.

Fitting a straight line by the method of least squares.

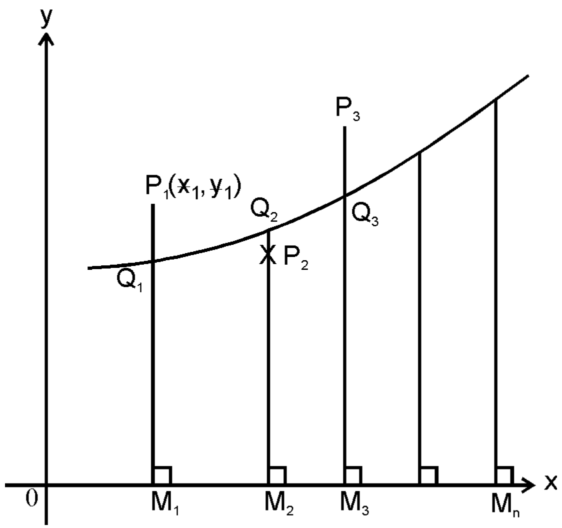
Let us consider the fitting a straight line y=a x + b …(1)
to the set of n points $(x_i, y_i), i = 1,2,3.........n$

For different values of a and b equation (1) represent a family of straight lines. Our aim is to determine a and b so that the line (1) is the line of 'best fit'

We apply the method of least squares to find the value of a and b. The principle of least square consist in minimizing the sum of the square of the deviations of the actual values y from its estimated values as given by the line of best fit.

Let $P_i(x_i, y_i)$ be any general point in the scatter diagram, $i = 1,2,3.........n$ in the n sets of observations and let

$$y=f(x) \qquad (1)$$



be the relation suggested between x and y. Let the ordinate at P_i meet y=f(x) at Q_i and the X axis at M_i

$$M_iQ_i = f(x_i), \text{and} M_iP_i = y_i$$

$$Q_iP_i = M_iP_i - M_iQ_i$$

$$= y_i - f(x_i), i = 1,2,3........, n$$

$d_i = y_i - f(x_i)$ is called the residual at $x = x_i$ Some of the d_i's may be positive and some may be negative

$E = \sum_{i=1}^{n} di^2 = \sum_{i=1}^{n} \left[y_i - f(x_i)^2\right]$ is the sum of the square of the residual.

If E=0, i.e., each d_i =0, Then all the n points P_i will lie on $y = f(x)$. If not, we will close $f(x)$ such that E is minimum. This principle is known as the as the principle of least squares.

The residual at $x = x_i$ is x_i

$$d_i = y_i - f(x_i) = y_i - (ax_i + b_i),\ i = 1,2,.....n$$

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$$

By the principle of least square, E is minimum

$$\frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0$$

$$\text{i.e.,} 2\sum [y_i - (ax_i + b)](-x_i) = 0 \text{ and } 2\sum [y_i - (ax_i + b)](-1) = 0$$

$$\sum_{i=1}^{n} \left(x_i y_i - a x_i^2 - b x_i\right) = 0 \qquad \text{and } \sum_{i=1}^{n} (y_i - ax_i - b) = 0$$

$$\text{i.e.,} \quad a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} xi = \sum_{i=1}^{n} x_i y_i \qquad (1)$$

$$a\sum_{i=1}^{n} x_i + nb = \sum_{i=1}^{n} y_i \qquad (2)$$

Since x_i, y_i are known, equations (1) & (2) give two equations in a and b. solve for a and b (1) and (2) and obtain the best fit $y = ax + b$.

Note

1. Equation (1) and (2) are called normal equation.
2. Dropping suffix i from (1) and (2), the normal equations are $a\sum x + nb = \sum y$ and $a\sum x^2 + b\sum x = \sum xy$ which are got by taking $\sum$ on both sides of y = ax +b and also taking $\sum$on both sides after multiplying by x both sides of y=ax+b

3. Transformations like $X = \frac{x-a}{h}, Y = \frac{y-b}{k}$ reduce the linear equations $y = \alpha x + \beta$ to the form y=Ax+B. Hence, a linear fit is another linear fit in both systems of co-ordinates.

3.3 WORKED EXAMPLES

PART – B

1). Fit straight line to following data by the method of least squares.

X :	5	10	15	20	25
Y:	15	19	23	26	30

Solution:

Let the straight line by Y= ax + b

The normal equations are

$$a\sum x + nb = \sum y$$
$$a\sum x^2 + b\sum x = \sum xy$$

To Calculate $\sum x, \sum x^2, \sum y, \sum xy$ we form below the table.

x	y	x^2	xy
5	16	25	80
10	19	100	190
15	23	225	345
20	26	400	520
25	30	625	750
75	114	1375	1885

The normal equations are

$$75a + 5b = 114 \qquad (1)$$

$$1375a + 75b = 1885 \qquad (2)$$

$(1)\times 15 \Rightarrow 1125a + 75b = 1710$

$(2)\times 1 \Rightarrow 1375a + 75b = 1885$

(3)-(4) $\Rightarrow$ 250a =175 or a=0.7

Hence b=12.3

Hence, the best fitting line is y=0.7x+12.3

ALITER:

Let $\Rightarrow X = \dfrac{x-15}{5}, Y = y - 23$

Let the line in the new variable be Y=AX+B

The normal equation are

$$A\sum X + nB = \sum Y$$
$$A\sum X^2 + B\sum X = \sum XY$$

x	y	X	X²	Y	XY
5	16	-2	4	-7	14
10	19	-1	1	-4	4
15	23	0	0	0	0
20	26	1	1	3	3
25	30	2	4	7	14
		0	10	-1	35

Substituting the values, we have

5B=-1

B=-0.2

$\therefore$ 10A=35 : A=3.5

The equation is y=3.5x-0.2

$$y - 23 = 3.5\left(\frac{x-15}{5}\right) - 0.2$$

=0.7x-10.5-0.2

y=0.7x+12.3

2.) Fit a straight line to the data given below. Also estimate the value of y at x=2.5

x:	0	1	2	3	4
y:	1	18	3.3	4.5	6.3

Solution:

Let the straight line be y = ax+b

The normal equation are

$$a\sum x + nb = \sum y$$

$$a\sum x^2 + b\sum x = \sum xy$$

To form the table:

x	y	x^2	x y
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
10	16.9	30	47.1

Substituting the values, we get

10a+56 = 16.9

30a+10B = 47.1

Solving, we get a=1.33, b=0.72

Hence, the equation of best fit is

y=1.33x+0.72

To find the value of y when x=2.5

y=1.33(2.5)+0.72

= 3.325+0.72

y=4.045

3.) Fit a straight line for the following data:

x:	0	12	24	36	48
y:	35	55	65	80	90

we form the table below:

Let $X = \frac{x-24}{12}$, $Y = \frac{y-65}{10}$

Let the line in the new variable be Y=AX+B The normal equation are $A\sum X + nB = \sum Y$; $A\sum X^2 + B\sum X = \sum XY$

x	y	X	X^2	Y	XY
0	35	-2	4	-3	6
12	55	-1	1	-1	1
24	65	0	0	0	0
36	80	1	1	1.5	1.5
48	90	2	4	2.5	5
		0	10	0	13.5

Substituting we values, we have

B=0

$\therefore$ A10 =13.5 ; A=1.35

The equation is Y=1.35X

$$\frac{y-65}{10} = 1.35\frac{(x-24)}{12}$$

$$y-65 = 13.5\frac{(x-24)}{12}$$

$$= 1.125x - 27$$

$y = 1.125x + 38$ is the equation of best fit.

4.) The following table shows the number of students in a post graduate course.

Year	1922	1993	1994	1995	1996
No.of Students	28	38	46	40	56

Use the method of least squares to fit a straight line trend and estimate the number of students in 1997.

Solution:

Let x denote the year and y the number of students

Let $X = x - 1994$ $\qquad Y = \frac{y-46}{2}$

Let the line of best fit be $Y = AX+B$

The normal equation are

$$A\sum X + nB = \sum Y$$

$$A\sum X^2 + B\sum X = \sum XY$$

The table is

x	y	X	Y	X^2	XY
1992	28	-2	-9	4	18
1993	38	-1	-4	1	4
1994	46	0	0	0	0
1995	40	1	-3	1	-3
1996	56	2	5	4	10
		0	-11	10	29

$\therefore$ we get , $0.A + 5B = -11$

$B = -2.2$

$10A - 0 = 29$ $\qquad A = 2.9$

$\therefore$ The line of best fit is $Y = 3X - 2.2$

$$\frac{y-46}{2} = 2.9(x-1994) - 2.2$$

$$y = 5.8(x-1994) + 41.6$$

The estimates of the number of students in 1997 is obtained on putting x=1997.

$Y = 5.8\ (1997 - 1994) + 41.6$

$\therefore y_{1997} = 5.8(3) + 41.6 = 59.0.$

5.) Fit a straight line trend to the following data

Year	1984	1985	1986	1987	1988	1989
Production (in tones)	7	9	12	15	18	23

Estimate the production for the year 1990.

Solution:

Let x denote the year and y the number of students

Let $X = x - 1987$

$Y = y - 15$

Let the line of best fit be $y = A x + B$

The normal equation are

$$A\sum X + nB = \sum Y$$

$$A\sum X^2 + B\sum X = \sum XY$$

To form the table:

x	y	X	Y	X^2	XY
1984	7	-3	-8	9	24
1985	9	-2	-6	4	12
1986	12	-1	-3	1	3
1987	15	0	0	0	0
1988	18	1	3	1	3
1989	23	2	8	4	16
		-3	-6	19	58

Substituting the values,

$$-3A + 6B = -6$$

$$19A - 3B = 58$$

Solving the equations, we get,

$$35A = 110$$

$$A = 3.142$$

$$-3(3.142) + 6B = -6$$

$$B = 0.571$$

$\therefore$ The eqn of best fit is

$$Y = 3.142\,X + 0.571$$

$$y - 15 = 3.142\,(x - 1987) + 0.571$$

The production for the year 1990 is

$$y = 3.142\,(1990 - 1987) + 15.571$$

$$= 3.142(3) + 15.571$$

$$= 9.426 + 15.571$$

$$= 24.997 \text{ tonnes.}$$

PART - A

1.) The Variance of a Poisson distribution is 0.35. Find P(X=2).

2.) For a Poisson distribution n =1000, λ=2 find 'p'

3.) In a Poisson distribution P(x=1) =P(x=2) Find λ

4.) If a random variable X- follows Poisson distribution such that P(x=2) =P(x=3). Find the mean of the distribution.

5.) If X is a Poisson distribution and P(X=0) = P(X=1) find the standard deviation.

6.) Write any two constant of Poisson distribution.

7.) Give any two examples of Poisson distribution.

8.) Under what conditions Poisson distribution is a good approximation of binomial distribution.

9.) Comment on the following “For a Poisson distribution mean =8 and valances =7”

10.) Define Poisson distribution

11.) Define normal distribution

12.) Define standard normal distribution

13.) Write down the constants of normal distribution

14.) Write down any three properties of normal distribution

15.) Write down the mean and standard deviation of the standard normal distribution

16.) Find the area that the standard normal variable lies between -1.56 and O from the table.

17.) Find the area to the right of 0.25

18.) Find the area to the left of z = 1.96

19.) Write down the normal equations for the straight line y= ax+b

20.) Write down the normal equations for the straight line y =a+bx

PART - B

1.) If X is a Poisson variable with P(X=2) $=\frac{2}{3}$ P(X=1) Find P(X=0) and P(X=3)

2.) 10% of the tools produced in a factory one found to be defective. Find the probability that in a set of 10 tools chosen at random exactly two will be defective.

3.) At a busy traffic junction, the probability p of an individual car having an accidents is 0.0001. However during certain part of the day 1000 car pass through the junction. What is the probability that two or more accidents occurs during that period.

4.) A telephone switch board receives on average of 5 emergency calls in a10 minute interval, what is the probability that (i) There are atmost 2 emergency call in ten minute interval.(ii) atleast 3 emergency call in a minute interval.

5.) A taxi firm has 2 cars which it hires out day by day. The number of demands of a car on each day is distributed as Poisson distribution with mean 1.5 Calculate the proposition of days on which.

(i) neither car is used (ii) some demand is refused.

6.) If 4% of the items manufactured by a Company are defective, find the probability that in a sample of 200 items (i) exactly one item is defective (ii) none is defective.

7.) The Probability of a Poisson variable taking the values 2 and 3 are equal, Calculate the valance and standard deviation.

8.) Articles of which 0.1 percent are defective are packed in boxes each containing 500 articles. (i) Using Poisson distribution find the probability that one box contains (i) no defective (ii) two or more defective articles $\left(e^{-0.5} = 0.6065\right)$

9.) A manufacture of pins known that 2% of his product are defective If he sells pins in boxes of 100 and guarantees that not more then 4 pins will be defective what is the probability that a box will fail to meet the guaranteed quantity.

10.) If a random variable X follows Poisson distribution, such that P(X=3) = P(X=2), find P(X=1).

11.) Find the mean and standard deviation of the normal distribution given by f(x) $= Ce^{\frac{1}{24}\left(x^2-6x+4\right)}$ $-\infty < x < \infty$

12.) Obtain the value of C, μ and σ^2 of the normal distribution whose probability density function is given by $f(x) = Ce^{-2x^2+4x}$ $-\infty < x < \infty$

13.) In America, a person travelled by jet plane may be affected by cosmic rays is normally distributed. Its mean is 4.35m rem and standard deviation is 0.59m rem. Find the probability for one person affected by cosmic rays above 5.20m rem.

14.) Students of a class were given an aptitude test. This marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored (i) more than60 marks (ii) lines than 56 marks (iii) between 45 and 65 marks.

15.) The life of a lamp produced by a factory is distributed normally with a mean of 50 days and standard deviation of 15 days. If 5000 lamps are fitted on the same day find the number of lamps to be replaced after 74 days.

16.) The life of automobile battery is normally distributed with mean 36 months and standard deviation of 5 months what is the probability that a particular battery last 28 to 44 months.

17.) The mean weight of 500 student is 68 kg and the standard deviation is 7kg. Assuming that the weight are normally distributed, find how many students weigh between 54kg and 75kg.

18.) In a normal distribution which is exactly normally 31% of the items are under 45 and 8% are over 64. Find the mean and the standard deviation of the distribution.

19.) In a normal distribution 7 percent of the items are below 35 and 11 percent of the items are above 63. Find the mean and standard deviation of the distribution.

20.) The mean weight of 500 students is 151 lb and the standard deviation is 151lb. Assuming that the weight are normally distributed, find (i) How may students weigh between 120 and 155 lb? (ii) How may weigh more than 185 lb.

21.) Fit a straight line to the following data

X	4	8	12	16	20	24
Y	12	15	19	22	26	30

22.) Fit a straight line for the following data by the method of least squares

X	0	1	2	3	4
Y	10	14	19	26	31

23.) Fit a straight line to the following data

X	1	2	3	4	5	6	7	8	9
Y	9	8	10	12	11	13	14	16	15

24.) Fit a straight line to the following data

X	2	4	6	8	10	12
Y	7	10	12	14	17	24

25.) Fit a straight line trend by the method of least squares to the following data. Also estimate the production for the year 1992.

Year	1985	1986	1987	1988	1989	1990
Production (Rs. In Crores)	7	10	12	14	17	24

26.) Fit a straight line to the following data

Year :	1960	1961	1962	1963	1964	1965	1966	1967
Value Find	380	400	650	720	690	600	870	930

Find the value for the year 1968.

27.) Fit a straight line to the following data

Year	1985	1986	1987	1988	1989
Sales	16	18	19	20	24

ANSWERS
PART - A

1.) $\dfrac{e^{-3.5}\lambda^2}{2}$ (2)P=0.002 (3) $\lambda = 2$

(4) mean=3 (5)SD=1 (9)wrong statement

15.) mean=0 SD=1 (16) 0.4406 (17) 0.4013

18.) 0.9750

19.) $\sum y = a\sum x + nb : \sum xy = a\sum x^2 = a\sum x^2 + b\sum x$

20.) $\sum y = na + b\sum x : \sum xy = a\sum x + b\sum x^2$

PART - B

1.) $\lambda = 1.3, e^{-1.3}, \frac{e^{-13}(1.3)^3}{6}$ **(2)** $\frac{1}{2e}$ **(3)** 0.0047

(4) (i) 0.1246 (ii) 0.8754

5.) (i) $e^{-1.5}$ (ii) $1 - e^{-1.5}(3.65)$

6.) (i) $8e^{-8}$ (ii) e^{-8}

7.) $\lambda = 3$

8.) (i) 0.6005 (ii) 0.0902

9.) 0.1429 (10) 0.1494

11.) $\mu = 3,\ \sigma = \sqrt{12},\ C = e^{-\frac{5}{24}} . \frac{1}{\sqrt{24\pi}}$

12.) $C = \sqrt{\frac{2}{\pi}} e^{-2},\ \mu = 1,\ \sigma^2 = \frac{1}{4}$

13.) 0.0749

14.) (i) 50% (ii) 21.19% (iii) 84%

15.) 0.0548 (16) 0.8904 (17) 409 students

18.) $\mu = 50,$ $\sigma = 10$

19.) $\mu = 50.3,$ $\sigma = 10.36$

20.) (i) 294 (ii) 6

21.) Y= (0.9) x+8.07 (22) Y=5.4x+9.2 (23)19x-20y+145=0

24.) Y=1.542x+26.794 (25) 27.86 Crores

26.) Value for the year 1968 =1124.162

27.) Y=1.8 (x-1987) + 19.4

POINTS TO REMEMBER

1.) Probability mass function : $P(x_i) \geq o$ for all x_i and $\sum_i p(xi) = 1$

2.) Probability density function : $f(x_i) \geq o$ for all x_i and $\int_{-\infty}^{\infty} f(x)dx = 1$

3.) Mean = E(X) = $\sum_{i=0}^{\infty} x_i \, P(x_i)$

4.) $E(X^2) = \sum_{i=0}^{\infty} x_i^2 P(x_i)$

5.) Variance of X= var (x)= E(x^2) - $[E(x)]^2$

6.) Binomial distribution : $P(x = x) = n_{c_x} P^x q^{n-x}$

n: no of trails
p= Prob of success
Q: Prob of failure

7.) Mean of binomial distribution = np
Variance = npq
S.D = $\sqrt{npq}$

8.) Poisson distribution $P(X = xi) = \frac{e^{-\lambda}\lambda^x}{x}$, x,0,1,2,3,.....

9.) Mean = variance = λ of a poisson distribution

10.) Pdf of normal distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$

11.) $z = \frac{x-\mu}{\sigma}$

12.) To fit the straight line y = ax+b, the normal equations are

$$\sum y = a\sum x + nb$$

$$\sum xy = a\sum x^2 + b\sum x$$

Standard Normal Distribution – Area Table

Z	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	**.1554**	**.1591**	**.1628**	**.1664**	**.1700**	**.1736**	**.1772**	**.1808**	**.1844**	**.1879**
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	**.3159**	**.3186**	**.3212**	**.3238**	**.3264**	**.3289**	**.3315**	**.3340**	**.3365**	**.3389**
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	**.4192**	**.4207**	**.4222**	**.4236**	**.4251**	**.4265**	**.4279**	**.4292**	**.4306**	**.4319**
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	**.4713**	**.4719**	**.4726**	**.4732**	**.4738**	**.4744**	**.4750**	**.4756**	**.4761**	**.4767**
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817

Standard Normal Distribution – Area Table

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	**.4918**	**.4920**	**.4922**	**.4925**	**.4927**	**.4929**	**.4931**	**.4932**	**.4934**	**.4936**
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4959	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	**.4981**	**.4982**	**.4982**	**.4983**	**.4984**	**.4984**	**.4985**	**.4985**	**.4986**	**.4986**
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
2.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	**.4997**	**.4997**	**.4997**	**.4997**	**.4997**	**.4997**	**.4997**	**.4997**	**.4997**	**.4998**
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**	**.5000**

UNIT- IV

APPLICATION OF INTEGRATION AND FIRST ORDER DIFFERENTIAL EQUATION

AREA AND VOLUME

4.1 Area – Area of circle, volume – volume of cone and sphere – simple problems

FIRST ORDER DIFFERENTIAL EQUATION

4.2 Definition of order and degree of differential equation – solution of first order variable separable type differential equation – simple problems

LINEAR TYPE DIFFERENTIAL EQUATION

4.3 Solution of linear differential equation – simple problems

Introduction

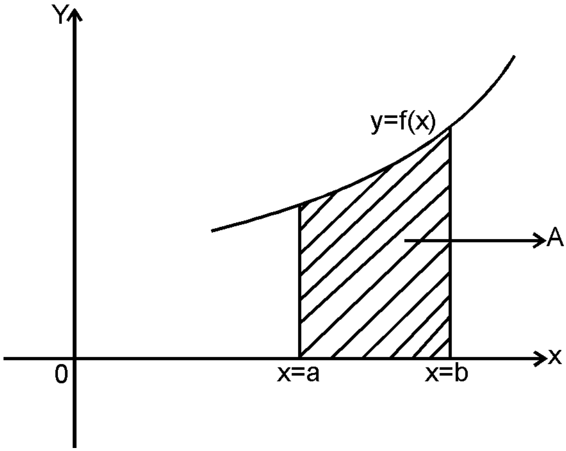
In mathematics – III, we discussed the basic concepts of integration. In mathematics – II, we studied the differential equation and formation of differential equation. In this unit, we shall study the application of integration and first order differential equation.

4.1. AREA AND VOLUME

We apply the concept of definite integral to find the area and volume.

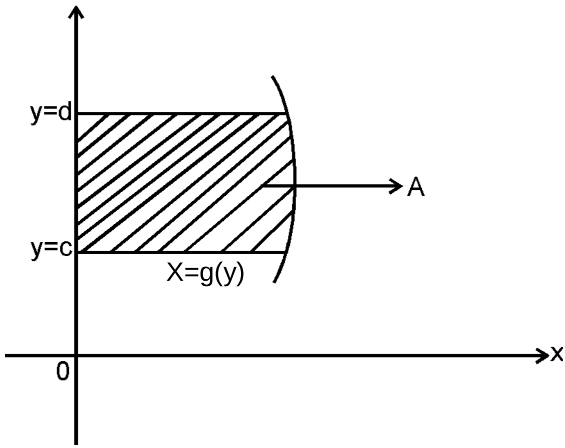
Area

The area under the curve y = f(x) between the x-axis and the ordinates x=a and x=b is the definite integral $\int_a^b f(x)dx$ or $\int_a^b ydx$



$$\text{Area(A)} = \int_a^b f(x)dx \quad \text{or} \quad \int_a^b ydx$$

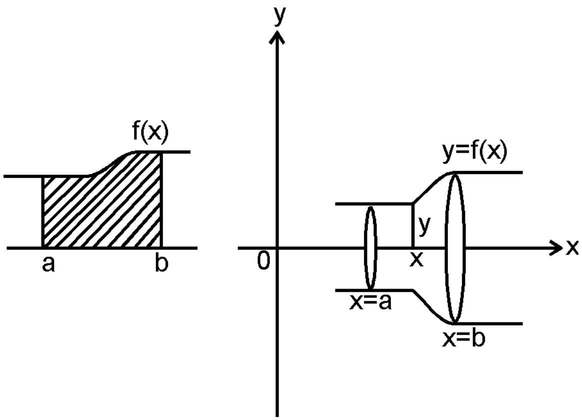
Similarly, the area between the curve x=g(y) and the y=axis and the lines y=c and y=d is the definite integral $\int_c^d g(y)dy$ or $\int_c^d xdy$



$$\text{Area (A)} = \int_c^d g(y)dy \quad \text{or} \quad \int_c^d xdy$$

Volume

The volume of solid obtained by rotating the area bounded by the curve y=f(x) and x-axis between x=a and x=b about the x-axis is the definite integral $\pi\int_a^b [f(x)]^2 dx$ or $\pi\int_a^b y^2 dx$



Volume (v)= $\pi\int_a^b [f(x)]^2 dx$ or $\pi\int_a^b y^2 dx$

Similarly, the volume of solid obtained by rotating the area bounded by the curve x = g(y) and y-axis between y=c and y=d about the y-axis is the definite integral $\pi\int_c^d [g(y)]^2 dy$ or $\pi\int_c^d x^2 dy$

Volume (v)= $\pi\int_c^d [g(y)]^2 dy$ or $\pi\int_c^d x^2 dy$

4.1 WORKED EXAMPLES

PART - A

1) Find the area bounded by the curve $y=4x^3$, the x-axis and the ordinates x=0 and x=1

Solution:

$$A = \int_a^b y\,dx = \int_0^1 4x^3 dx = \left[4\frac{x^4}{4}\right]_0^1$$

$$= \left[x^4\right]_0^1 = (1)^4 - 0 = 1 - 0 = 1 \text{ Sq. units}$$

2) Find the area bounded by the curve $y=e^x$, the x-axis and the ordinates x=0 and x=6.

Solution:

$$A = \int_a^b y\,dx = \int_0^6 e^x dx$$

$$= \left[e^x\right]_0^6 = e^6 - e^0 = \left(e^6 - 1\right) \text{Sq. units}$$

3) Find the area bounded by the curve y=cosx, x-axis and between x=0 and x=$\frac{\pi}{2}$

Solution:

$$A = \int_a^b y\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 \text{Sq. units}$$

4) Find the area bounded by the curve $x=4y-y^2$, the y-axis and the lines y=0 and y=3

Solution:

$$A = \int_c^d x\,dy \quad = \int_0^3 \left(4y - y^2\right) dy$$

$$= \left[\frac{4y^2}{2} - \frac{y^3}{3}\right]_0^3 = \left[2y^2 - \frac{y^3}{3}\right]_0^3$$

$$= 18 - 9 = 9 \text{ Sq. units}$$

5) Find the volume of the solid formed when the area bounded by the curve $y^2=4x$ between x=1 and x=2 is rotated about x-axis.

Solution:

$$v = \pi\int_a^b y^2 dx \quad = \pi\int_1^2 4x dx$$

$$= 4\pi\left[\frac{x^2}{2}\right]^2$$

$$= 4\pi\left[\frac{(2)^2}{2} - \frac{(1)^2}{2}\right]$$

$$= 4\pi.\frac{3}{2}$$

$$= 6\pi \text{ Cubic units}$$

6) Find the volume of the solid formed when the area bounded by the curve y=$\sqrt{10+x}$ between x=0 and x=5 is rotated about x-axis.

Solution:

Given $y = \sqrt{10+x}$

Squaring on both sides, we get

$y^2=10+x$

$$v = \pi\int_a^b y^2 dx \quad = \pi\int_0^5 (10+x)dx$$

$$= \pi\left[10x + \frac{x^2}{2}\right]_0^5$$

$$= \pi\left[50 + \frac{25}{2}\right]$$

$$= \frac{125}{2}\pi \text{ Cubic units.}$$

7) Find the volume of the solid formed when the area bounded by the curve $x^2 = \sec^2 y$ between y=0 and $y = \frac{\pi}{4}$ is rotated about y-axis.

Solution:

$$v = \pi \int_c^d x^2 dy \quad = \pi \int_0^{\frac{\pi}{4}} \sec^2 y dy$$

$$= \pi [\tan y]_0^{\frac{\pi}{4}}$$

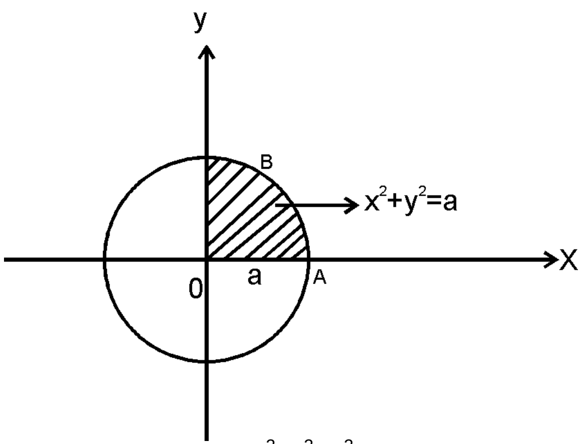
$$= \pi \left[\tan \frac{\pi}{4} - \tan 0 \right] = \pi [1 - 0]$$

$$= \pi \text{ Cubic units.}$$

PART - B

1. Find the area of the circle whose radius is 'a' units.

Solution:



Area of AOB bounded by $x^2+y^2=a^2$, the x-axis between x=0 and x=a is

$$\text{Area of } \quad AOB = \int_a^b y\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \left[\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right) - (0+0)\right]$$

$$= \frac{a^2}{2}.\sin^{-1}(1)$$

$$= \frac{a^2}{2}.\frac{\pi}{2} = \frac{\pi a^2}{4}$$

$\therefore$ Required area $= 4 \times$ Area of AOB

$$= 4 \times \frac{\pi a^2}{4}$$

$\therefore$ Area of circle $= \pi a^2$ Squnits.

Aliter

$$I = \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta)\,d\theta \qquad \text{Put } x = a\sin\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - \sin^2\theta)}\,(a\cos\theta)\,d\theta \qquad dx = a\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{a^2\left(1-\sin^2\theta\right)}\left(a\cos\theta\right)d\theta \qquad \text{when } x = 0,\ \theta = 0$$

$$= \int_0^{\frac{\pi}{2}} a^2\cos\theta\cos\theta d\theta \qquad \text{and } x = a,\ \theta = \frac{\pi}{2}$$

$$= a^2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta \quad \left(\therefore \cos^2 A = \frac{1+\cos 2A}{2}\right)$$

$$= a^2\left[\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}(0) - 0 - \frac{1}{2}(0)\right] = \frac{\pi a^2}{4}$$

$$\therefore \text{Area of the circle} = 4\frac{\pi a^2}{4} = \pi a^2$$

2. Find the area bounded by the curve $y=x^2+x+2$, the x-axis and the ordinate $x=1$ and $x=2$.

Solution:

$$A = \int_a^b y dx = \int_1^2 \left(x^2 + x + 2\right)dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_1^2$$

$$= \left[\frac{2^3}{3} + \frac{2^2}{2} + 2(2)\right] - \left[\frac{1^3}{3} + \frac{1^2}{2} + 2(1)\right]$$

$$= \left(\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} + 2\right)$$

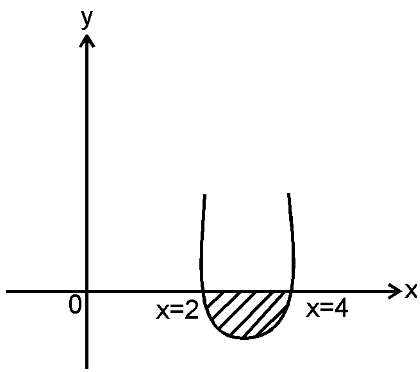
$$= \left(\frac{8}{3} + \frac{6}{1}\right) - \left(\frac{2+3+12}{6}\right)$$

$$= \frac{26}{3} - \frac{17}{6} = \frac{52-17}{6}$$

$$= \frac{35}{6} \text{ Sq units}$$

3. Find the area bounded by the curve $y=x^2-6x+8$ and the x-axis.

Solution:



The curve meets the x-axis at y=0

$y = 0 \Rightarrow x^2-6x+8=0$

$\Rightarrow (x-2)(x-4)=0$

$\Rightarrow x=2,4$

$$A = \int_a^b y\,dx = \int_2^4 (x^2 - 6x + 8)dx = \left(\frac{x^3}{3} - 6\frac{x^2}{2} + 8x\right)_2^4$$

$$= \left(\frac{x^3}{3} - 3x^2 + 8x\right)_2^4$$

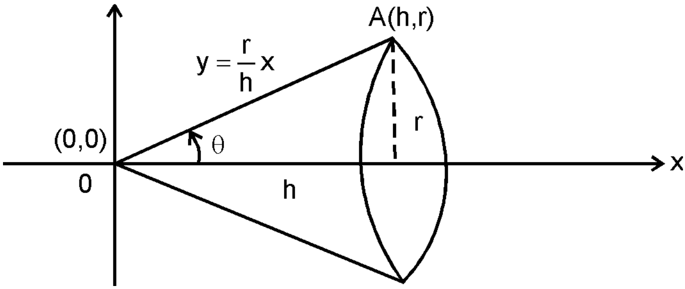
$$= \left[\left(\frac{64}{3} - 48 + 32\right) - \left(\frac{8}{3} - 12 + 16\right)\right]$$

$$= \frac{64}{3} - 16 - \frac{8}{3} - 4 = \frac{64}{3} - \frac{8}{3} - 20 = \frac{64 - 8 - 60}{3}$$

$$= \frac{-4}{3} = \frac{4}{3} \quad \text{Sq units.}$$

4. Find the volume of a right circular cone of base radius r and altitude h by integration.

Solution:



Let y=mx be rotated about the x-axis to get the cone.

Then $m = \tan\theta = \frac{r}{h}$.

Now $y = mx \Rightarrow y = \frac{r}{h}x$

Volume $V = \pi\int_a^b y^2 dx = \pi\int_0^h \left(\frac{r}{h}x\right)^2 dx$

$$= \frac{\pi r^2}{h^2}\int_0^h x^2 dx$$

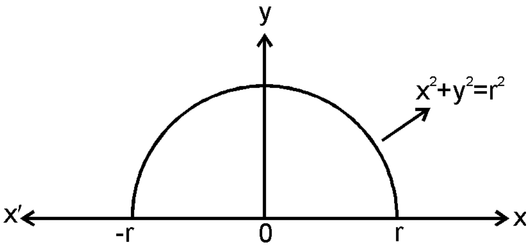
$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\left[\frac{h^3}{3} - 0\right]$$

$$V = \frac{\pi r^2 h^3}{h^2} = \frac{1}{3}\pi r^2 h \text{ Cubic units.}$$

5. Find the volume of the sphere of radius r by integration.

Solution:

When the semi-circle $x^2 + y^2 = r^2$ is rotated about X-axis solid sphere is obtained.



Volume of the sphere $V = \pi \int_a^b y^2 dx$

$$= \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx$$

$$= 2\pi \int_0^r \left(r^2 - x^2\right) dx \qquad (\because \text{function is even})$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$= 2\pi \left[r^3 - \frac{r^3}{3} - (0-0) \right]$$

$$= 2\pi . \frac{2r^3}{3} \; v = \frac{4}{3}\pi r^3 \text{cubic units.}$$

6. Find the volume of the solid generated when the region enclosed by $y^2=4x^3+3x^2+2x$ between x=1 and x=2 is revolved about x-axis.

Solution:

$$v = \pi\int_a^b y^2 dx$$

$$= \pi\int_1^2 \left(4x^3+3x^2+2x\right)dx = \left[4\frac{x^4}{4}+3\frac{x^3}{3}+2\frac{x^2}{2}\right]_1^2$$

$$= \pi\left[x^4+x^3+x^2\right]_1^2$$

$$= \pi\left[\left(2^4+2^3+2^2\right)-\left(1^4+1^3+1^2\right)\right]$$

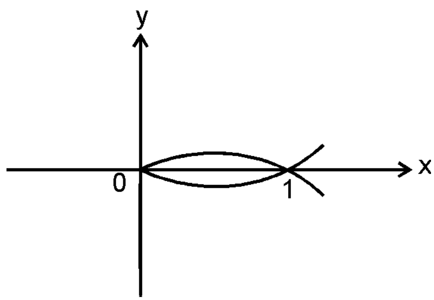
$$= \pi\left[(16+8+4)-(1+1+1)\right]$$

$$= \pi\left[(28-3)\right]$$

$$= 25\pi \text{ Cubic units.}$$

7. Find the volume generated by the area enclosed by the curve $y^2=x(x-1)^2$ and the x-axis, when rotated about x-axis.

Solution:



The curve meets x-axis

$\therefore y=0$

$\Rightarrow y^2=0$

$\Rightarrow x(x-1)^2=0$

$\Rightarrow x=0,1$

$$v = \pi\int_a^b y^2 dx$$

$$= \pi\int_0^1 x(x-1)^2 dx$$

$$= \pi\int_0^1 x\left(x^2 - 2x + 1\right)dx$$

$$= \pi\int_0^1 \left(x^3 - 2x^2 + x\right)dx$$

$$= \pi\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right) - (0 - 0 + 0)\right]$$

$$= \pi\left[\frac{3-8+6}{12}\right] = \pi\left[\frac{1}{12}\right] = \frac{\pi}{12} \text{ Cubic units.}$$

4.2. FIRST ORDER DIFFERENTIAL EQUATION

Introduction:

Since the time of Newton, physical problems have been investigated by formulating them mathematically as differential equations Many mathematical models in engineering employ differential equations extensively.

Order and degree of Differential Equation:

The order of a differential equation is the order of the highest differential coefficient appearing in the equation.

The degree of an equation is the degree of the highest differential coefficient is free from radicals and fractional exponents.

Solution of First Order Differential Equation:

Recalling that we formed differential equation by differentiating algebraic equations involving x,y etc and constants. Now, we will consider the reverse process.

Consider the differential equation

$$\frac{dy}{dx} = 3x^2 \qquad \ldots(1)$$

A solution of this equation is $y=x^3$, since this satisfy (1) All the Possibilities which satisfy are of the form

$$y = x^3 + c \qquad \ldots(2)$$

This is called the general solution of (1), here c may be any constant. We call c an arbitrary constant. The general solution of (1) has one arbitrary constant. For second order differential equation, two arbitrary constants will be there.

A particular solution is one where a value is given to c. Particular solutions arise when we are required to find a solution fitting certain conditions.

If we give y=1 when x=0 in (2), we get c=1.

Hence the particular solution is $y=x^3+1$.

Solution of the variable separable differential equation:

In the first order differential equation, say $\frac{dy}{dx} = f(x, y)$, if the function of x can be grouped with dx on one side and the function of y can be grouped with dy on the other side, then this type of equation is called variable separable differential equation. The solution can be obtained by integrating both sides after separating the variables.

4.2 WORKED EXAMPLES

PART - A

1. Write the order and degree of the following differential equations

i $y = 2\frac{dy}{dx} + 3\frac{d^2y}{dx^2} + 5$

ii $y' + y^2 = 0$

Iii $(D^2+5D+4)y=e^x$

iv $\frac{d^2y}{dx^2} = 3x = \sqrt{\frac{dy}{dx} + 2y}$

Solution:

i. $y = 2\frac{dy}{dx} + 3\frac{d^2y}{dx^2} + 5$

Here, Highest order=2

Degree of highest order=1

$\therefore$Order=2, degree=1

ii. $y' + y^2 = 0 \quad \Rightarrow \frac{dy}{dx} + y^2 = 0$

Here, Highest order=1

Degree of Highest order=1

$\therefore$Order=1, degree=1

iii $(D^2 + 5D + 4)y = e^x$

$$\Rightarrow \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = e^x$$

Here, Highest order=2

Degree of Highest order=1

$\therefore$Order=2, degree=1

iv $\dfrac{d^2y}{dx^2}+3x=\sqrt{\dfrac{dy}{dx}+2y}$

To eliminate the radical in the above equation, raising to the power 2 on both sides, we get $\left(\dfrac{d^2y}{dx^2}+3x\right)^2=\dfrac{dy}{dx}+2y$

Here, Highest order=2

Degree of Highest order=2

$\therefore$ Order=2, degree=2

2. Solve $x^8dx+y^8dy=0$

Solution:

Given $x^8dx+y^8dy=0$

Integrating, we get, $\int x^8dx+\int y^8dy=0$

$$\frac{x^9}{9}+\frac{y^9}{9}=c$$

3. Solve $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$

Solution:

Given $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$

$$\Rightarrow \quad dy=\frac{2}{1+x^2}dx$$

Integrating, we get, $\int dy=\int\dfrac{2dx}{1+x^2}$

$$y=2\tan^{-1}x+c$$

4. Solve $\frac{dy}{dx}=\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$

Solution:

Given $$\frac{dy}{dx}=\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{dx}{\sqrt{1-x^2}}$$

Integrating, we get $\int\frac{dy}{\sqrt{1-y^2}}=\int\frac{dx}{\sqrt{1-x^2}}$

$$\text{Sin}^{-1}y=\sin^{-1}x+c$$

5. Solve $\frac{dy}{dx}=\frac{3+x}{3+y}$

Solution:

Given $$\frac{dy}{dx}=\frac{3+x}{3+y}$$

$$\Rightarrow (3+y)dy=(3+x)dx$$

Integrating, we get, $\int(3+y)dy=\int(3+x)dx$

$$3y+\frac{y^2}{2}=3x+\frac{x^2}{2}+c$$

6. Solve $\frac{dy}{dx} = e^{x-5y}$

Solution:

Given $$\frac{dy}{dx} = e^{x-5y}$$

$$\Rightarrow \frac{dy}{dx} = e^{x}e^{-5y}$$

$$\Rightarrow \frac{dy}{e^{-5y}} = e^{x}dx$$

$$\Rightarrow e^{5y}dy = e^{x}dx$$

Integrating, we get $\int e^{5y}dy = \int e^{x}dx$

$$\frac{e^{5y}}{5} = e^{x} + c$$

7. Solve $x\frac{dy}{dx} = y$

Solution:

Given $$x\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Integrating, we get, $\int \frac{dy}{y} = \int \frac{dx}{x}$

$$\log y = \log x + \log c$$

$$\Rightarrow \log y = \log cx$$

$$\Rightarrow y = cx$$

8. Find the solution of $\Rightarrow \frac{dy}{dx} - y\cos x = 0$

Solution:

Given $\frac{dy}{dx} - y\cos x = 0$

$$\Rightarrow \frac{dy}{dx} = y\cos x$$

$$\Rightarrow \frac{dy}{y} = \cos x dx$$

Integrating, we get, $\int \frac{dy}{y} = \int \cos x dx$

$$\text{Log} y = \sin x + c$$

PART - B

1) Solve $(x^2 - y)\, dx + (y^2 - x)\, dy = 0$

Solution:

Given $(x^2 - y)\, dx + (y^2 - x)\, dy = 0$

$$\Rightarrow x^2dx - ydx + y^2dy - xdy = 0$$

$$\Rightarrow x^2dx + y^2dy = xdy + ydx$$

$$\Rightarrow x^2dx + y^2dy = d(xy)$$

Integrating, we get $\int x^2 dx + \int y^2 dy = \int d(xy)$

$$\frac{x^3}{3} + \frac{y^3}{3} = xy + c$$

2) Solve $\frac{dy}{dx}+\frac{1+\cos 2y}{1+\cos 2x}=0$

Solution:

Given $\frac{dy}{dx}+\frac{1+\cos 2y}{1+\cos 2x}=0$

$$\Rightarrow \frac{dy}{dx}=-\frac{1+\cos 2y}{1+\cos 2x}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2\cos^2 y}{2\cos^2 x}$$

$$\Rightarrow \frac{dy}{\cos^2 y}=-\frac{dx}{\cos^2 x}$$

$$\Rightarrow \sec^2 ydy=-\sec^2 xdx$$

Integrating, we get, $\int \sec^2 ydy=-\int \sec^2 xdx$

tany = -tanx + c

$\Rightarrow$tanx+tany=c

3) Solve $\frac{dy}{dx}=e^{x-y}+3x^2e^{-y}$

Solution:

Given $\frac{dy}{dx}=e^{x-y}+3x^2e^{-y}$

$$\Rightarrow \frac{dy}{dx}=e^xe^{-y}+3x^2e^{-y}$$

$$\Rightarrow \frac{dy}{dx}=e^{-y}\left[e^x+3x^2\right]$$

$$\Rightarrow e^ydy=\left[e^x+3x^2\right]dx$$

Integrating, we get. $\int e^ydy=\int\left(e^x+3x^2\right)dx$

$e^y = e^x + x^3 + c$

4) Solve $\frac{dy}{dx} = \frac{y^2 + 4y + 5}{x^2 - 2x + 2}$

Solution:

Given $$\frac{dy}{dx} = \frac{y^2 + 4y + 5}{x^2 - 2x + 2}$$

$$\Rightarrow \frac{dy}{y^2 + 4y + 5} = \frac{dx}{x^2 - 2x + 2}$$

$$\Rightarrow \frac{dy}{(y+2)^2 + 1} = \frac{dx}{(x-1)^2 + 1^2}$$

Integrating, we get $\int \frac{dy}{(y+2)^2 + 1^2} = \int \frac{dx}{(x-1)^2 + 1^2}$

$$\tan^{-1}(y+2) = \tan^{-1}(x - 1) + c$$

5) Solve $(1-e^x)\sec^2 y dy + e^x \tan y \, dx = 0$

Solution:

Given $(1-e^x)\sec^2 y dy + e^x \tan y \, dx = 0$

$$\Rightarrow (1-e^x)\sec^2 y dy = -e^x \tan y \, dx$$

$$\Rightarrow \frac{\sec^2 y dy}{\tan y} = \frac{-e^x dx}{1 - e^x}$$

Integrating, we get $\int \frac{d(\tan y)}{\tan y} = \int \frac{d(1 - e^x)}{1 - e^x}$

$\log(\tan y) = \log(1-e^x) + \log c$

$$\Rightarrow \log(\tan y) = \log c(1-e^x)$$

$$\Rightarrow \tan y = c(1-e^x)$$

4.3. LINEAR TYPE DIFFERENTIAL EQUATION

A first order differential equation is said to be linear in y if the power of the terms $\frac{dy}{dx}$ and y are unity.

$\frac{dy}{dx}+Py=Q$ is a linear differential equation. Here P and Q are function of x. The solution of linear differential equation is given by $ye^{\int pdx} = \int Qe^{\int pdx}dx + c$

i.e. $\quad y\,(IF) = \int Q(IF)\,dx + c$

Here IF = $e^{\int pdx}$ is called an Integrating factor

Note:

$$e^{\log f(x)} = f(x)$$

4.3 WORKED EXAMPLES

PART - A and B

1.) Find the integrating factor of $\frac{dy}{dx}+\frac{5}{x}y = x$

Solution:

Given $\frac{dy}{dx}+\frac{5}{x}y = x$

Here $P = 5/x$

$$IF = e^{\int pdx} = e^{5\int \frac{1}{x}dx}$$

$$= e^{5\log x}$$

$$= e^{\log x^5}$$

$$IF = x^5$$

2.) Find the integrating factor of $\frac{dy}{dx} - x - y = 0$

Solution:

Given $\frac{dy}{dx} - x - y = 0$

$$\Rightarrow \frac{dy}{dx} - y = x$$

Here P=-1

IF = $e^{\int pdx} = e^{\int -dx} = e^{-x}$

3.) Find the integrating factor of

$$\frac{dy}{dx} - \sin 2x = y \cot x$$

Solution:

Given $\frac{dy}{dx} - \sin 2x = y \cot x$

$$\Rightarrow \frac{dy}{dx} - y \cot x = \sin 2x$$

Here P = $-\cot x$

$$IF = e^{\int pdx}$$

$$= e^{-\int \cot x d x}$$

$$= e^{-\log \sin x}$$

$$= e^{\log \operatorname{cosec} x}$$

$$IF = \operatorname{cosec} x$$

PART - B

1.) Solve $\frac{dy}{dx} + 3y = \frac{1}{3}$

Solution:

Given $\frac{dy}{dx} + 3y = \frac{1}{3}$

Here P = 3 and Q = $\frac{1}{3}$

Now, IF= $e^{\int p dx}$

$$= e^{3\int dx} = e^{3x}$$

The Required solution is

$$y \times IF = \int Q \times IF\ dx + c$$

$$y e^{3x} = \int \frac{1}{3} e^{3x} dx + c$$

$$\Rightarrow y e^{3x} = \frac{e^{3x}}{9} + c$$

2.) Solve $\frac{dy}{dx} + \frac{3x^2 y}{1+x^3} = \frac{2}{1+x^3}$

Here $P = \frac{3x^2}{1+x^3}$ and $Q = \frac{2}{1+x^3}$

Now, IF $= e^{\int p dx}$

$$= e^{\int \frac{3x^2}{1+x^3} dx}$$

$$= e^{Log(1+x^3)}$$

$$IF = 1 + x^3$$

The Required solution is

$$y(IF) = \int Q(IF) dx + c$$

$$y(1+x^3) = \int \frac{2}{1+x^3}(1+x^3) dx + c$$

$$y(1+x^3) = 2\int dx$$

$$y(1+x^3) = 2x + c$$

3.) Solve $\frac{dy}{dx} + y\cot x = 2\cos x$

Solution:

Given $\frac{dy}{dx} + y\cot x = 2\cos x$

Here P = Cotx , Q= 2 Cos x

Now, If = $e^{\int pdx}$

$$= e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

The Required Solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y\sin x = \int 2\cos x \sin x dx + c$$

$$\Rightarrow y\sin x = \int \sin 2x dx + c$$

$$\Rightarrow y\sin x = -\frac{\cos 2x}{2} + c$$

4.) Solve $\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$

Solution:

Given $\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$

Here $P = -3/x, Q = x^3 \cos x$

Now, $IF = e^{\int pdx}$

$$= e^{\int -3\frac{1}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

The Required solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y\frac{1}{x^3} = \int x^3 \cos x \frac{1}{x^3} dx + c$$

$$\Rightarrow \frac{y}{x^3} = \int \cos x dx + c$$

$$\Rightarrow \frac{y}{x^3} = \sin x + c$$

5.) Solve $\left(1+x^2\right)\frac{dy}{dx}+2xy=1$

Solution:

Given $\left(1+x^2\right)\frac{dy}{dx}+2xy=1$

Divide both sides by $\left(1+x^2\right)$, we get

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{1+x^2}$$

Here P= $\frac{2xy}{1+x^2}$, $\quad Q=\frac{1}{1+x^2}$

Now If $= e^{\int pdx}$

$$= e^{\frac{2x}{1+x^2}dx}$$

$$= e^{\log\left(1+x^2\right)}$$

$$= 1+x^2$$

The Required solution is

$$y(IF) = \int Q(IF)dx + c$$

$$y\left(1+x^2\right) = \int \frac{1}{1+x^2}\left(1+x^2\right)dx + c$$

$$\Rightarrow y\left(1+x^2\right) = \int dx + c$$

$$\Rightarrow y\left(1+x^2\right) = x + c$$

EXERCISE

PART - A

1. Find the area bounded by the curve y= 2x the x-axis and the ordinates x =0 and x=1.
2. Find the are bounded by the curve $y=x^2$ and x-axis between x=0 and x=2
3. Find the area bounded by the curve $y=\frac{x^2}{2}$, x-axis, and between x=1, and x=3.

4. Find the area under the curve $y = \frac{1}{1+x^2}$ x- axis,

$x = -1$ and $x = 1$

5. Find the area bounded by the curve y=sinx, x-axis and between x=0 and x = π

6. Find the area bounded by the curve $y^2=3x$, the x-axis and line x=3

7. Find the area bounded by the curve xy=1 the y-axis and the lines y=1 and y=5.

8. Find the area bounded by the curve x=2y+5, the y-axis and the lines y=1 and y=2.

9. Find the volume of the solid formed when the area bounded by the curve $y^2=25x^3$ between x =1 and x =3is rotated about x- axis.

10. Find the volume of the solid formed when the area bounded by the curve $y^2=8x$ between x=0 and x=2 rotated about x-axis.

11. Find the volume generated by rotating the triangle with vertices at (0,0),(3,0) and (3,3) about x-axis.

12. Find the volume generated when the area bounded by the curve $x^2=3y^2$ between y=0 and y=1 is rotated about y-axis.

13. Write the order and degree of the following differential equations.

(i) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + y = e^x$

(ii) $y^{11} + y^2 = 0$

(iii) $y'' = \frac{dy}{dx} + \frac{dx}{dy}$

(iv) $\frac{d^2y}{dx^2} = \left(1 + \frac{dy}{dx}\right)^{1/3}$

14. Solve xdx+ydy=0

15. Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$

16. Solve $\frac{dy}{dx} = 3x^2y$

17. $\frac{dy}{dx} = \frac{1+x}{1+y}$

18. $\frac{dy}{dx} = xy + y + x + 1$

19. Solve $\frac{dy}{dx} = e^{2x-y}$

20. Solve $\frac{dy}{dx} = \frac{-y}{x}$

21. Find the solution of $\frac{dy}{dx} + y\sin x = 0$

22. Find the integrating factor of $\frac{dy}{dx} + \frac{1}{x}y = x$

23. Find the integrating factor of $\frac{dy}{dx} + \frac{1}{1+x^2}y = 1$

24. Find the integrating factor of $\frac{dy}{dx} - y\tan x = e^x \sec x$

25. Find the integrating factor of $2\cos x\frac{dy}{dx} + 4y\sin x = \sin 2x$

PART - B

1) Find the area of the circle whose radius is 4 units using integration.

2) Find the area of region bounded by the curve y= $3x^2 - 4x + 5$, the x-axis and the lines x=1 and x=2

3) Find the area bounded by the curve $y = x^2 + x + 1$ and x-axis and the ordinates x=1 and x=3

4) Find the area bounded by the curve $y = 4x - x^2$ and the x-axis

5) Find the area bounded by the curve $y = 10 - 3x - x^2$ and the x-axis

6) Find the volume of the solid formed when the area bounded by the curve $y^2 = 2 + x - x^2$ the x-axis and the lines x=-1 and x=2.

7) Find the volume of the solid formed when the area bounded by the curve $x^2 = \frac{a^2}{b^2}\left(b^2 - y^2\right)$, the y-axis and the lines y=-b and y=b

8) Solve $\left(xy^2 + x\right)dx + \left(yx^2 + y\right)dy = 0$

9) Solve $\frac{dy}{dx} = \frac{1+\cos y}{1+\cos x}$

10) Solve $\frac{dy}{dx} = e^{x+y} + xe^{y}$

11) Solve $\frac{dy}{dx} = e^{3x-2y} + x^3e^{-2y}$

12) Solve $\frac{dy}{dx} = \frac{y^2 + 2y + 10}{x^2 + 2x + 10}$

13) Solve $\left(1 + x^2\right)\sec^2 y\,dy = 2x\tan y\,dx$

14) Solve $\left(1 + e^x\right)\sec^2 y\,dy - e^x \tan y\,dx = 0$

15) Solve $3e^x \tan y\,dx + \left(1 + e^x\right)\sec^2 y\,dy = 0$

16) Solve $\left(e^x + 1\right)\cos y\,dy + e^x \sin y\,dx = 0$

17) Solve $\frac{dy}{dx} + 3y = 6$

18) Solve $\frac{dy}{dx} + \frac{y}{x} = x^4$

19) Solve $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{1+x^2}$

20) Solve $\frac{dy}{dx} + y = x$

21) Solve $\frac{dy}{dx} + y\cot x = e^{x} \operatorname{cosec} x$

22) Solve $\frac{dy}{dx} + y\tan x = 4x\cos x$

23) Solve $\frac{dy}{dx} - \frac{2y}{x} = x^{2}\sin x$

24) Solve $x\frac{dy}{dx} - 3y = x^{4}e^{x}$

25) Solve $(1+x^{4})\frac{dy}{dx} + 4x^{3}y = \frac{1}{1+x^{4}}$

ANSWERS
PART - A

1) 1 Sq unit

2) $\frac{8}{3}$ Sq unit

3) $\frac{13}{3}$ Sq unit

4) $\frac{\pi}{2}$ Sq unit

5) 2 Sq unit

6) 6 Sq unit

7) log 5 Sq unit

8) 8 Squnit

9) 500 π cubic unit

10) 16 π Cubic units

11) 9 π cubic units

12) π cubic units

13) (i) order =3, degree=1 (ii) order =2, degree=1
(iii) order =1, degree=2 (iv) order =2, degree=3

14) $\frac{x^{2}}{2} + \frac{y^{2}}{2} = c$

15) $\frac{y^{3}}{3} - \frac{x^{3}}{3} = c$

16) $\log y = x^{3} + c$

17) $y + \frac{y^{2}}{2} = x + \frac{x^{2}}{2} + c$

18) $\operatorname{Log}(1+y) = \frac{x^{2}}{2} + x + c$

19) $e^y = \frac{e^{2x}}{2} + c$ 20) xy=c 21) logy=cosx+c

22) X (23) $e^{\tan^{-1}x}$ (24) Cos x (25) $\sec^2 x$

PART - B

1) 16π Sq. units (2) 6 Sq. units (3) $\frac{44}{3}$ Sq. units

4.) $\frac{32}{3}$ Sq. units (5) $\frac{343}{6}$ Sq. units (6) $\frac{9\pi}{2}$ Cubic units

7) $\frac{4}{3}\pi a^2 b$ cubic units (8) $\sqrt{x^2+1}\sqrt{y^2+1} = c$

9) $\tan\frac{y}{2} = \tan\frac{x}{2} + c$ (10) $e^x + e^{-y} + \frac{x^2}{2} = c$

11) $\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^4}{4} + c$ (12) $\tan^{-1}\left(\frac{y+1}{3}\right) - \tan^{-1}\left(\frac{x+1}{3}\right) = c$

13) $\tan y = c(1+x^2)$ 14) $\tan y = c(1+e^x)$

15) $\tan y\,(1+e^x)^3 = c$ 16) $(e^x+1)\sin y = c$

17) $ye^{3x} = 2e^{3x} + c$ (18) $xy = \frac{x^6}{6} + c$

19) $y(1+x^2) = x + c$ (20) $ye^x = xe^x - e^x + c$

21) $y\sin x = e^x + c$ (22) $y\sec x = 2x^2 + c$

23) $\frac{y}{x^2} + \cos x = c$ (24) $\frac{y}{x^3} = e^x + c$

25) $y(1+x^4) = x + c$

UNIT - V

SECOND ORDER DIFFERENTIAL EQUATIONS

5.1. Solution of second order differential equations with constant coefficients in the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$ Simple Problems

5.2 Solution of second order differential equations in the form $a\ \frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$. Where a,b and c are constants and $f(x) = e^{mx}$. Simple problems.

5.3. Solution of second order differential equations in the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x).$ Where a,b and c are constants and f(x) = sinmx or cosmx. Simple problems

5.1 SECOND ORDER DIFFERENTIAL EQUATIONS

Introduction:

In the last unit, we learnt first order differential equation. In this unit, we will learn second order differential equation.

The second order differential equation is of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x). \qquad (1)$$

Where a,b and c are real numbers and f(x) is a function of x.

We use differential operators Dy, D^2y in (1), we get

$$(aD^2+bD+c)y=f(x) \text{ where } D=\frac{d}{dx} \qquad (2)$$

Now, we put f(x) = 0 in (1), we get

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad (3)$$

The solution of (3) is called complementary function (CF) of (1).

To solve (3), we assume a trial solution $y = e^{px}$ for some value of p. Then $\frac{dy}{dx} = pe^{px}$ and $\frac{d^2y}{dx^2} = p^2e^{px}$.

Substituting these values in (3), we get

$$ap^2e^{px} + bpe^{px} + ce^{px} = 0$$

$$\Rightarrow e^{px}[ap^2 + bp + c] = 0 \qquad (4)$$

$$\Rightarrow ap^2 + bp + c = 0$$

This equation in p is called the Auxillary Equation (AE)

Solving (4), we get two roots say p_1 and p_2. Then the following three cases arise.

Case (i)

If the roots p_1 and p_2 are real and distinct, then the solution of (3) is

$$y = Ae^{p_1x} + Be^{p_2x}$$

Case (ii)

If the roots p_1 and p_2 are real and equal, then the solution of (3) is

$$y = e^{p_1x}(Ax + B)$$

Case (iii)

If the roots p_1 and p_2 are complex say $p_1 = \alpha + i\beta$ and $p_2 = \alpha - i\beta$, then the solution of (3) is

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

In all cases, A and B are arbitrary constants.

5.1 WORKED EXAMPLES

PART – A

1. If roots of the auxillary equation are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, what is the solution of the differential equation?

Solution:

Here, the roots are complex and $\alpha = \frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$,

$\therefore$ The solution of differential equation is

$$y = e^{\frac{1}{2}x}[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x]$$

2. Find the solution of $(D^2 - 81)\, y = 0$

Solution:

The auxillary equation is $p^2 - 81 = 0$

$\Rightarrow (p+9)\,(p-9) = 0$

$\Rightarrow p_1 = -9,\ p_2 = 9$

Here, the roots are real and distinct

$\therefore$ The solution of differential equation is

$$y = Ae^{-9x} + Be^{9x}$$

3. Solve $\frac{d^2y}{dx^2} + 64y = 0$

Solution:

Given $\frac{d^2y}{dx^2} + 64y = 0 \Rightarrow (D^2 + 64)y = 0$

The auxillary equation is $p^2+64=0$

$\Rightarrow p = \pm 8i$

Here, the roots are complex, $\alpha=0$ and $\beta = 8$

$\therefore$ The solution is $y = A\cos 8x + B\sin 8x$

4. Solve $(D^2-2D-3)y=0$

Solution:

The auxillary equation is $p^2-2p-3=0$

$$\Rightarrow (p+1)(p-3) = 0$$

$$\Rightarrow p_1 = -1, p_2 = 3$$

Here, the roots are real and distinct

$\therefore$ The solution is $y = Ae^{-x} + Be^{3x}$

5. Solve $(D^2-4D-1)\, y = 0$

Solution:

The auxillary equation is $p^2-4p-1 = 0$

Here a = 1, b = -4, c = -1

$$P = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-4(1)(-1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{20}}{2}$$

$$= \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

So, $p_1 = 2 + \sqrt{5}$ and $p_2 = 2 - \sqrt{5}$

Here, the roots are real and distinct

$\therefore$ The solution is

$y = Ae^{(2+\sqrt{5})x} + Be^{(2-\sqrt{5})x}$

6. Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

Solution:

Given: $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0 \;\Rightarrow\; (D^2-6D+9)\,y=0$

The auxillary equation is $p^2 - 6p+9 = 0$

$\Rightarrow (p-3)(p-3)=0$

$\Rightarrow p_1=3,\ p_2 = 3$

Here, the roots are real and equal.

.. The solution is $y = e^{3x}[Ax+B]$

7. Solve $(D^2+D+2)y=0$

Solution:

The auxillary equation is $p^2+p+2 = 0$

Here a = 1, b = 1, c = 2

$$P = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1\pm\sqrt{1-4(1)(2)}}{2(1)}$$

$$= \frac{-1\pm\sqrt{-7}}{2}$$

$$= \frac{-1\pm i\sqrt{7}}{2}$$

$$= \frac{-1}{2}\pm i\frac{\sqrt{7}}{2}$$

Here, the roots are complex, $\alpha=-\frac{1}{2},\beta=\frac{\sqrt{7}}{2}$

$\therefore$ The solution is $y = e^{\frac{-1}{2}x}[A\cos\frac{\sqrt{7}}{2}x+B\sin\frac{\sqrt{7}}{2}x$

PART – B

1. Solve $(D^2+1)\,y = 0$ when $x = 0$, $y = 2$ and $x = \frac{\pi}{2}$, $y=-2$.

Solution:

The auxillary equation is $p^2 + 1 = 0$

$$\Rightarrow p = \pm i$$

Here, the roots are complex, $\beta = 1$

$\therefore$ The solution is

$y = A\cos x + B\sin x$ …1

When x=0, y=2, the equation (1) becomes

$A\cos 0 + B\sin 0 = 2$

$A + 0 = 2$

$A = 2$

When $x = \frac{\pi}{2}$, y=-2, the equation (1) becomes

$A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2} = -2$

$0 + B = -2$

$B = -2$

$\therefore$ The required solution is

$y = 2\cos x - 2\sin x$

2. Show that the solution of the equation $(D^2 + 3D + 2)\,y = 0$ if $y(0) = 1$ and $y^1(0) = 0$ is $y = 2e^{-x} - e^{-2x}$

Solution:

The auxillary equation is $p^2+3p+2=0$

$$\Rightarrow (p+1)(p+2) = 0$$

$$\Rightarrow p_1 = -1,\ p_2 = -2$$

Here, the roots are real and distinct

$\therefore$ The solution is $y = Ae^{-x} + Be^{-2x}$ …1

Now, $y' = -Ae^{-x} - 2Be^{-2x}$ …2

If y(0) = 1, the equation (1) becomes

$A + B = 1$ …3

If y'(0) =0 , the equation (2) becomes

$A+2B=0$ …4

Solving (3) and (4) we get A=2, B=-1

$\therefore$The required solution is

$y=2e^{-x}-e^{-2x}$

5.2. SOLUTION OF SECOND ORDER EQUATIONS IN THE FORM $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$ WHERE A,B AND C ARE CONSTANTS AND $f(x)=e^{mx}$.

Introduction:

In previous section, we find the complementary function . In this section, we have to find the particular integral (PI) and the general solution of a second order differential equation.

The Solution of Differential equation with Constant Coefficients is y=CF+PI

Method of finding particular integral

Consider $(aD^2+bD+c)y = e^{mx}$ where m is a constant.

Let $f(D) = aD^2+bD+c$

Then PI is given by $\frac{1}{f(D)}e^{mx} = \frac{e^{mx}}{f(m)}$

Three cases arise in PI

Case (i)

If $f(m) \neq 0$ then $PI = \frac{1}{f(D)}e^{mx} = \frac{e^{mx}}{f(m)}$

Case (ii)

$$\text{If } f(m)=0 \text{ and } f'(m)\neq 0 \text{ then} \quad PI=\frac{x\,e^{mx}}{f'(m)}$$

Case (iii)

$$\text{If } f(m)=0 \text{ and } f'(m)=0 \text{ and } f''(m)\neq 0 \text{ then} \quad PI=\frac{x^2 e^{mx}}{f''(m)}$$

5.2 WORKED EXAMPLE

PART – A

1. Find the complementary function of $(D^2+16)y= e^x$

Solution:

The auxiliary equation is $p^2+16=0 \Rightarrow p=\pm 4i$

Here, the roots are complex , $\beta=4$

$\therefore$ CF = A cos 4x + B sin 4x

2. Find the complementary function of $(D^2-60D+800)y=e^{40x}$

Solution:

The auxiliary equation is $p^2-60p+800=0$

$\Rightarrow (p-40)\,(P-20)=0$

$\Rightarrow P_1=40,\ P_2=20$

Here the roots are real and distinct

$\therefore CF = Ae^{40x} + Be^{20x}$

3. Find the particular integral of $(D^2+1)\,y=1$

Solution:

$$PI=\frac{1}{D^2+1}=\frac{1}{D^2+1}\,e^0$$

$$=\frac{1}{0+1}=\frac{1}{1}=1$$

4. Find the particular integral of $(D^2+7D+14) = 8e^{-x}$

Solution:

$$PI = \frac{1}{D^2+7D+14} 8e^{-x}$$

$$= \frac{8e^{-x}}{(-1)^2+7(-1)+14} = \frac{8e^{-x}}{8} = e^{-x}$$

5. Find the particular integral of $(D^2-2D-3)y = e^{-x}$

Solution:

$$PI = \frac{1}{D^2-2D-3} e^{-x}$$

$$= \frac{x\,e^{-x}}{2D-2} \qquad \text{Since } f(-1)=0$$

$$= \frac{x\,e^{-x}}{2(-1)-2} = -\frac{x\,e^{-x}}{4}$$

PART - B

1. Solve $(D^2+5D+6)y=30$

Solution:

The auxiliary equation is $p^2+5p+6=0$

$\Rightarrow (p+2)\,(P+3) = 0$

$\Rightarrow P_1=-2,\ P_2=-3$

Here, the roots are real and distinct

$\therefore CF = Ae^{-2x}+Be^{-3x}$

$$\text{Now } PI = \frac{1}{D^2+5D+6} 30$$

$$= \frac{30e^{\circ}}{D^2+5D+6}$$

$$= \frac{30e^{\circ}}{0^2+5(0)+6}$$

$$= \frac{30}{6}$$

$$PI = 5$$

$\therefore$ The Required solution is

$Y=CF+PI = Ae^{-2x}+Be^{-3x}+5$

2. Solve $(D^2+6D+5)\ y = 2e^x$

Solution:

The auxiliary equation is $p^2+6p+5=0$

$\Rightarrow (p+1)\ (P+5) = 0$

$\Rightarrow P_1=-1,\ P_2=-5$

Here the roots are real and distinct

$\therefore CF = Ae^{-x} + Be^{-5x}$

$$\text{Now PI} = \frac{1}{D^2+6D+5}\ 2e^x$$

$$= \frac{2e^x}{1^2+6(1)+5}$$

$$= \frac{2e^x}{12}$$

$$PI = \frac{e^x}{6}$$

$\therefore$ The required solution is

$Y=CF+PI$

$$Ae^{-x} + Be^{-5x} + \frac{e^x}{6}$$

3. Solve $(D^2+D)y = e^{\frac{x}{2}}$

Solution:

The auxiliary equation is $p^2+p=0$

$\Rightarrow p\ (p+1)=0$

$\Rightarrow P_1=0,\ P_2=-1$

Here the roots are real and distinct

$\therefore CF = Ae^0 + Be^{-x} = A+Be^{-x}$

$$\text{Now PI} = \frac{1}{D^2+D}\ e^{\frac{x}{2}}$$

$$= \frac{e^{\frac{x}{2}}}{\left(\frac{1}{2}\right)^2 + \frac{1}{2}}$$

$$= \frac{e^{\frac{x}{2}}}{3/4}$$

$$PI = \frac{4}{3} e^{\frac{x}{2}}$$

$\therefore$ The required solution is

y=CF+PI

$$= A + Be^{-x} + \frac{3}{4} e^{\frac{x}{2}}$$

4. Solve $(D^2 - D - 12)y = e^{4x}$

Solution:

The auxiliary equation is $p^2-p-12=0$

$\Rightarrow (p-4)(p+3)=0$

$\Rightarrow p_1=4,\ p_2=-3$

Here the roots are real and distinct

$\therefore CF = Ae^{4x} + Be^{-3x}$

$$\text{Now } PI = \frac{1}{D^2 - D - 12} e^{4x}$$

$$= \frac{x e^{4x}}{2D - 1} \qquad \text{Since } f(4) = 0$$

$$= \frac{x e^{4x}}{2(4) - 1}$$

$$PI = \frac{x e^{4x}}{7}$$

$\therefore$ The required solution is

y=CF + PI

$$= Ae^{4x} + Be^{-3x} + \frac{x e^{4x}}{7}$$

5. Solve $(D^2-2D+1)\ y = e^x$

Solution:

The auxiliary equation is $p^2-2p+1=0$

$\Rightarrow (p-1)\ (p-1) = 0$

$\Rightarrow p_1=1,\ p_2=1$

Here the roots are real and equal

$\therefore CF = e^x\ (Ax+B)$

$$\text{Now}\ PI = \frac{1}{D^2-2D+1}\ e^x$$

$$PI = \frac{x^2}{2}\ e^x \qquad \text{Since}\ \ f(1)=0, f'(1)=0$$

$\therefore$ The required solution is

Y=CF+PI

$$= e^x\left(Ax+B\right)+\frac{x^2}{2}e^x$$

6 Solve $\dfrac{d^2y}{dx^2}-13\dfrac{dy}{dx}+12y = 2e^{-2x}+5e^x$

Solution:

$$\text{Given}\quad \frac{d^2y}{dx^2}-13\frac{dy}{dx}+12y = 2e^{-2x}+5e^x$$

$$\Rightarrow (D^2-13D+12)\,y = 2e^{-2x}+5e^x$$

The auxilary equation is $p^2-13p+12=0$

$\Rightarrow (p-1)\ (p-12) = 0$

$\Rightarrow p_1=1,\ p_2=12$

Here the roots are real and distinct

$\therefore CF = Ae^x+Be^{12x}$

$$\text{Now } PI_1 = \frac{1}{D^2 - 13D + 12}\, 2e^{-2x}$$

$$= \frac{2e^{-2x}}{(-2)^2 - 13(-2) + 12}$$

$$= \frac{2e^{-2x}}{4 + 26 + 12}$$

$$= \frac{e^{-2x}}{21}$$

$$\text{Now } PI_2 = \frac{1}{D^2 - 13D + 12}\, 5e^{x}$$

$$= \frac{5xe^{x}}{2D - 13} \qquad \text{Since } f(1) = 0$$

$$= \frac{5xe^{x}}{2(1) - 13}$$

$$= -\frac{5xe^{x}}{11}$$

$\therefore$ The required solution is

Y =CF+PI_1+PI_2

$$= Ae^{x} + Be^{12x} + \frac{e^{-2x}}{21} - \frac{5xe^{x}}{11}$$

5.3 SOLUTION OF SECOND ORDER DIFFERENTIALEQUATIONS IN THE FORM $a\frac{d^2y}{dx^2} - b\frac{dy}{dx} + cy = f(x)$ WHERE a,b AND c ARE CONSTANTS AND f(x) = sin mx or cos mx where m is a constant ≠ 0

INTRODUCTION

In this section, we have to find the particular integral when f(x) =sin mx or cos m x where m is a constant

Methods of finding PI

Consider f(x) =sin m x

Case (i)

Express f(D) as function of D^2,say ϕ (D^2) and then replace $D^2$with $-m^2$

If $\phi(-m^2)\neq 0$,then

$$PI = \frac{1}{f(D)}\sin mx$$

$$= \frac{1}{\phi(D^2)}\sin mx$$

$$PI = \frac{1}{\phi(-m^2)}\sin mx$$

Case (ii)

Sometimes we cannot form ϕ (D^2) Then we shall try to get $\phi(D,D^2)$ that is a function of D and D^2. In such cases we proceed as follows.

For Example

$$\text{Now PI} = \frac{1}{D^2+2D+3}\sin 2x$$

$$= \frac{1}{-2^2+2D+3}\sin 2x \text{ Replace } D^2 \text{ by } -2^2$$

$$= \frac{1}{2D-1}\text{Sin } 2x$$

$$= \frac{2D+1}{4D^2-1}\sin 2x \text{ multiply and divide by } 2D+1$$

$$= \frac{2D(\sin 2x)+\sin 2x}{4(-2^2)-1}$$

$$= \frac{4\cos 2x+\sin 2x}{-17}$$

$$= \frac{1}{-17}[4\cos 2x + \sin 2x]$$

Now consider f(x) = cos m x

Case (i): $PI = \frac{1}{\phi(-m^2)}\cos mx$

Case(ii): Same as sin m x method

General Solution:

The general solution is y= CF+PI

5.3 WORKED EXAMPLE

PART - A

1. Find the complementary function of (D^2+49) y= cos 4x

Solution:

The auxiliary equation is $p^2+49=0$

$\Rightarrow p=\pm 7i$

Here, the roots are complex ,β =7

$\therefore$ CF = A cos 7x+B sin 7x

2. Find the particular integral of (D^2+14) y = sin 3x

Solution:

$$PI = \frac{1}{D^2+14}\sin 3x$$

$$= \frac{1}{-3^2+14}\sin 3x$$

$$= \frac{\sin 3x}{5}$$

3. Find the particular integral of (D^2+a^2) y = Cos b x

Solution:

$$PI = \frac{1}{D^2+a^2}\cos bx$$

$$= \frac{1}{-b^2+a^2}\cos bx$$

$$= \frac{\cos bx}{a^2-b^2}$$

PART - B

1.) Solve $(D^2-4)y = \sin 2x$

Solution:

The auxiliary equation is $p^2-4=0$

$$\Rightarrow p^2 = 4$$

$$\Rightarrow p = \pm 2$$

$$\Rightarrow p_1 = 2, p_2 = -2$$

Here, the roots are real and distinct

$$\therefore\ CF = Ae^{2x} + Be^{-2x}$$

$$\text{Now } PI = \frac{1}{D^2-4}\left(\sin 2x\right)$$

$$= \frac{1}{-2^2-4}\sin 2x$$

$$= -\frac{\sin 2x}{8}$$

$\therefore$ The Required solution is

$$y = CF + PI$$

$$= Ae^{2x} + Be^{-2x} - \frac{\sin 2X}{8}$$

2.) Solve $D^2y = -16\sin 4x$

Solution:

The auxiliary equation is $p^2 = 0$

$$\Rightarrow p_{,} = 0, p_2 = 0$$

Here, the roots are real and equal

$$\therefore\ CF = e^0\left(Ax + B\right) = Ax + B$$

$$\text{Now } PI = \frac{1}{D^2} - 16\sin 4x$$

$$= \frac{1}{-4^2} - 16\sin 4x$$

PI= Sin4x

$\therefore$ The Required solution is

$$y = CF + PI$$

$$= Ax + B + Sin4x$$

3.) Solve $\frac{d^2y}{dx^2}+16y=\cos^2 x$

Solution:

Given $\frac{d^2y}{dx^2}+16y=\cos^2 x$

$$\Rightarrow (D^2+16)y=\cos^2 x$$

$$\Rightarrow (D^2+16)y=\frac{1}{2}+\frac{\cos 2x}{2}$$

$$=\frac{1}{2}e^0+\frac{1}{2}\cos 2x$$

The auxiliary equation is $p^2+16=0$

$$\Rightarrow p=\pm 4i$$

Here, the roots are complex, $\beta=4$

$$\therefore CF = A\cos 4x + BSin4x$$

$$PI_1=\frac{\frac{1}{2}e^0}{D^2+16}$$

$$=\frac{1}{2}\,\frac{e^0}{0+16}$$

$$=\frac{1}{32}$$

$$PI_2=\frac{1}{2}\cdot\frac{\cos 2x}{D^2+16}$$

$$=\frac{1}{2}\cdot\frac{\cos 2x}{-2^2+16}$$

$$=\frac{\cos 2x}{24}$$

$\therefore$ The Required solution is

$$y=CF+PI$$

$$=A\cos 4x+BSin4x+\frac{1}{32}+\frac{\cos 2x}{24}$$

4.) Solve $(D^2 + 3D + 2)y = \sin 2x$

Solution:

The auxiliary equation is $p^2 + 3p + 2 = 0$

$$\Rightarrow (p+2)(p+1) = 0$$

$$\Rightarrow p_1 = -2,\ p_2 = -1$$

Here, the roots are real and distinct

$\therefore CF = Ae^{-2x} + Be^{-x}$

$$\text{Now, } PI = \frac{1}{D^2 + 3D + 2}.\text{Sin}2x$$

$$= \frac{1}{-2^2 + 3D + 2}.\text{Sin}2x$$

$$= \frac{1}{3D - 2}.\text{Sin}2x$$

$$= \frac{3D + 2}{9D^2 - 4}.\text{Sin}2x$$

$$= \frac{3D + 2}{-36 - 4}.\text{Sin}2x$$

$$= \frac{3D(\sin 2x) + 2\sin 2x}{-40}$$

$$= \frac{6\cos 2x + 2\sin 2x}{-40}$$

$$= \frac{-1}{20}[3\cos 2x + \sin 2x]$$

$\therefore$ The Required solution is

$y = CF + PI$

$$= Ae^{-2x} + Be^{-x} - \frac{1}{20}[3\cos 2x + \sin 2x]$$

5.) Solve $(D^2-2D-8)y=4\cos 3x$

Solution:

Solution: The auxiliary equation is $p^2-2p-8=0$

$$\Rightarrow (p-4)(p+2)=0$$
$$\Rightarrow p_1=4,\ p_2=-2$$

Here, the roots are real and distinct

$\therefore CF = Ae^{4x}+Be^{-2x}$

Now, $PI=\dfrac{1}{D^2-2D-8}4\cos 3x$

$$=\frac{1}{-3^2-2D-8}4\cos 3x$$

$$=\frac{1}{-2D-17}4\cos 3x$$

$$=-4\left[\frac{1}{2D+17}4\cos 3x\right]$$

$$=-4\left[\frac{2D-17}{4D^2-289}\cos 3x\right]$$

$$=-4\left[\frac{2D(\cos 3x)-17\cos 3x}{-325}\right]$$

$$=-4\left[\frac{-6\sin 3x-17\cos 3x}{-325}\right]$$

$$=\frac{-4}{325}[6\sin 3x+17\cos 3x]$$

$\therefore$ The Required solution is

$y=CF+PI$

$$=Ae^{4x}+Be^{-2x}-\frac{4}{325}[6\sin 3x+17\cos 3x]$$

EXERCISE

PART - A

1.) If roots of the auxilary equation are 2,7 what is the solution of the differential equation?

2.) If roots of the auxilary equation are 0,1 what is the solution of the differential equation?

3.) If roots of the auxilary equation are -2,$\pm$ i, what is the solution of the differential equation?

4.) Find the solution of $(D^2 - 1)y = 0$

5.) Find the solution of $\frac{d^2y}{dx^2} - 16y = 0$

6.) Solve $(D^2 + 9)y = 0$

7.) Find the solution of $(D^2 + 100)y = 0$

8.) Solve $(D^2 + 4D - 1020)y = 0$

9.) Solve $(3D^2 - 5D + 2)y = 0$

10.) Solve $(3D^2 - 7D - 6)y = 0$

11.) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

12.) Solve $(D^2 - D - 1)y = 0$

13.) Solve $(D^2 + 4D + 4)y = 0$

14.) Solve $\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 36y = 0$

15.) Solve $(D^2 + D + 1)y = 0$

16.) Solve $(3D^2 - D + 1)y = 0$

17.) Find the Complementary function of $(D^2 + 13D - 90)y = e^x$

18.) Find the Particular integral of $(D^2-3D+2)y=e^{-x}$

19.) Find the Particular integral of $(D^2+D+4)y=10e^{2x}$

20.) Find the Particular integral of $(D^2-8D+15)y=e^{3x}$

21.) Find the Particular integral of $(D^2+10D+25)y=e^{-5x}$

22.) Find the Complementary integral of $(D^2+25)y=\cos ax$

23.) Find the Particular integral of $(D^2+25)y=\text{Sin}x$

24.) Find the Particular integral of $(D^2+10)y=\sin 3x$

25.) Find the Particular integral of $\frac{d^2y}{dx^2}-4y=\cos 4x$

PART - B

1.) Solve $(D^2+36)y=0$ when $y(0)=2$ and $y^1(0)=12$

2.) Solve $\frac{d^2y}{dx^2}+y=0$ given that $\frac{dy}{dx}=2$ and y=1 when x=0

3.) Solve $(D^2-2D-15)y=0$ given that $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=2$ when x=0

4.) Solve $(D^2-D-20)y=0$ given that y=5 and $\frac{dy}{dx}=-2$ when x=0

5.) Solve $(D^2+7D+12)y=3$

6.) Solve $(D^2+3D+2)y=2e^x$

7.) Solve $(D^2+12D+36)y=e^x$

8.) Solve $(D^2+D+4)y=e^{x/2}$

9.) Solve $(D^2-3D+2)y=e^{2x}$

10.) Solve $(D^2+6D+8)y=e^{-4x}$

11.) Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$

12.) Solve $(D^2 + 2aD + a^2)y = e^{-ax}$

13.) Solve $(D^2 + 14D + 49)y = 4e^{-7x}$

14.) Solve $(D^2 - 2D + 4)y = 5 + 3e^{-x}$

15.) Solve $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = e^{-3x} + e^{3x}$

16.) Solve $(D^2 + 10D + 25)y = e^{5x} + e^{-5x}$

17.) Solve $(D^2 + 16)y = \sin 9x$

18.) Solve $(D^2 - 25)y = \sin 5x$

19.) Solve $(D^2 + 100)y = \cos 2x$

20.) Solve $\frac{d^2y}{dx^2} - 2y = \cos 3x$

21.) Solve $(D^2 + 2D - 3)y = \sin x$

22.) Solve $(D^2 + D - 2)y = \mathrm{Sin} 3x$

23.) Solve $(D^2 + 4D + 13)y = 4\cos 3x$

24.) Solve $(D^2 - 8D + 9)y = 8\cos 5x$

25.) Solve $(D^2 - 2D - 8)y = 4\cos 2x$

ANSWERS
PART - A

1.) $y = Ae^{2x} + Be^{7x}$

2.) $y = A + Be^{x}$

3.) $y = e^{-2x}[A\cos x + B\sin x]$

4.) $y = Ae^{x} + Be^{-x}$

5.) $y = Ae^{4x} + Be^{-4x}$

6.) $y = A\cos 3x + B\sin 3x$

7.) $y = A\cos 10x + B\sin 10x$

8.) $y = Ae^{30x} + Be^{-34x}$

9.) $y = Ae^{x} + Be^{\frac{2}{3}x}$

10.) $y = Ae^{3x} + Be^{-\frac{2}{3}x}$

11.) $y = A + Be^{-x}$

12.) $y = Ae^{\left(\frac{1+\sqrt{5}}{2}\right)x} + Be^{\left(\frac{1-\sqrt{5}}{2}\right)x}$

13.) $y = e^{-2x}(Ax + B)$

14.) $y = e^{6x}(Ax + B)$

15.) $y = e^{-x/2}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$

16.) $y = e^{x/6}\left(A\cos\frac{\sqrt{11}}{6}x + B\sin\frac{\sqrt{11}}{6}x\right)$

17.) $CF = Ae^{5x} + Be^{-18x}$

18.) $\frac{e^{-x}}{6}$

19.) e^{2x}

20.) $-\frac{xe^{3x}}{2}$

21.) $\frac{x^2}{2}e^{-5x}$

22.) CF=Acos5x+Bsin5x

23.) $\frac{\sin x}{24}$

24.) Sin3x

25.) $-\frac{\text{Cos}4x}{20}$

Part - B

1.) $y = 2\cos 6x + 2\sin 6x$

2.) $y = \cos x + 2\sin x$

3.) $y = \frac{1}{20}e^{5x} + \frac{1}{2}e^{-3x}$

4.) $y = 2e^{5x} + 3e^{-4x}$

5.) $y = Ae^{-4x} + Be^{-3x} + \frac{1}{4}$

6.) $y = Ae^{-x} + Be^{-2x} + \frac{e^{x}}{3}$

7.) $y = e^{-6x}(Ax + B) + \frac{e^{x}}{49}$

8.) $y = e^{-x/2}\left[A\cos\frac{\sqrt{15}}{2}x + B\sin\frac{\sqrt{15}}{2}x\right] + \frac{4}{19}e^{x/2}$ 9.)

$y = Ae^{x} + Be^{2x} + xe^{2x}$

10.) $y = Ae^{-4x} + Be^{-2x} - \frac{xe^{-4x}}{2}$

11.) $y = e^{2x}(Ax + B) + \frac{x^2}{2}e^{2x}$

12.) $y = e^{-ax}(Ax + B) + \frac{x^2}{2}e^{-ax}$

13.) $y = e^{-7x}(Ax + B) + 2x^2e^{-7x}$

14.) $y = e^{x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + \frac{5}{4} + \frac{3}{7}e^{-x}$

15.) $y = Ae^{-3x} + Be^{-5x} + \frac{xe^{-3x}}{2} + \frac{e^{3x}}{48}$

16.) $y = e^{-5x}(Ax + B) + \frac{e^{5x}}{100} + \frac{x^2e^{-5x}}{2}$

17.) $y = A\cos 4x + B\sin 4x - \frac{\sin 9x}{65}$

18.) $y = Ae^{5x} + Be^{-5x} - \frac{\sin 5x}{50}$

19.) $y = A\cos 10x + B\sin 10x + \frac{\cos 2x}{96}$

20.) $y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} - \frac{\cos 3x}{11}$

21.) $y = Ae^{-3x} + Be^{x} - \frac{1}{10}(\cos x + 2\sin x)$

22.) $y = Ae^{x} + Be^{-2x} - \frac{1}{130}(3\cos 3x + 11\sin 3x)$

23.) $y = e^{-2x}(A\cos 3x + B\sin 3x) + \frac{1}{10}3\sin 3x + \cos 3x$

24.) $y = Ae^{(4+\sqrt{7})x} + Be^{(4-\sqrt{7})x} - \frac{1}{29}(5\sin 5x + 2\cos 5x)$

25.) $y = Ae^{4x} + Be^{-2x} - \frac{1}{10}(\sin 2x + 3\cos 2x)$

MODEL QUESTION PAPER – 1

ENGINEERING MATHEMATICS- IV

Time : 3 Hrs **Max Marks : 75**

PART – A

(Marks: 15 x 1 = 15)

Answer any fifteen (15) questions:

1. Find the value of $i^2 + i^3 + i^4$
2. If $z_1 = 1 + i$, $z_2 = 3 + 2i$ the find $3z_1+4z_2$.
3. Find the modulus and amplitude of $\frac{1}{2} + i\frac{\sqrt{3}}{2}$
4. Find the distance between the complex numbers 2 + I and 1 – 2i.
5. Find the value of $(\cos\theta + i\sin\theta)^2 (\cos 3\theta + i\sin 3\theta)^{-3}$
6. If x= $(\cos\theta + i\sin\theta)$, what is the value of $x^m + 1/x^m$
7. If ω is a cube root of unity, then find the value of $1 + \omega^2 + \omega^4$.
8. Simplify $(1+ \omega)(1+ \omega^2)$
9. If the mean of the Poisson distribution is 2, find P(X=0).
10. Give two examples of Poisson distribution.
11. State the normal distribution.
12. Write down the normal equations to fit a straight line y = ax+b.
13. Find the area bounded by the curve $y = x^2$ and x-axis between x = 0 and x = 2.
14. Solve xdx + ydy = 0.
15. Find the solution of $\frac{dy}{dx} + y\sin x = 0$
16. Find the integrating factor of $\frac{dy}{dx} - y\cot x = \sin x$
17. Find the solution of $(D^2 – 1)y = 0$
18. Find the complementary function of $(D^2 + 1)y = e^{2x}$
19. Find the particular integral of $(D^2 + 5D + 6)y = 13$
20. Find the auxiliary equation of $(D^2 + 9)y = \sin 4x$

PART - B

(Marks: 5 x 12 = 60)

[N.B :- (1) Answer all questions choosing any two divisions from each question.

(2) All questions carry equal marks.]

21 (a) Find the real part and imaginary part of the complex number $\dfrac{(1+i)(2-i)}{1+3i}$

(b) Find the modulus and amplitude of the complex number $\dfrac{1+\sqrt{3}i}{1+i}$

(c) Show that the complex numbers (2 –2i), (8 +4i), (5+7i), (-1 + i) form a rectangle.

22 (a) Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)^2 \ (\cos 3\theta - i\sin 3\theta)^4}{(\cos 3\theta + i\sin 3\theta)^2 \ (\cos 4\theta + \sin 4\theta)^{-2}}$

(b) If *n* is a positive integer, prove that

$$\left(\sqrt{3}+I\right)^n - \left(\sqrt{3}-i\right)^n = 2^{n+1} \cos\frac{n\pi}{6}$$

(c) Solve: $x^7 + 1 = 0$

23 (a) In a Poisson distribution if P(X=3) = P(X=2) find P(X=0) and P(X=1).

(b) If X is normally distributed with mean 80 and standard deviation 10 find $P(70 \leq x \leq 100)$.

(c) Fit a straight line for the following data.

X	0	1	2	3	4
Y	10	14	19	26	31

24 (a) Find the volume of a right circular cone of base radius r and altitude h by Integration.

(b) Solve $\frac{dy}{dx}+\frac{1+\cos 2y}{1+\cos 2x}=0$

(c) Solve $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$

25 (a) Solve : (D^2 + 36)y = 0 when x=0 , y=2 and when x=$\frac{\pi}{2}$, y = 3

(b) Solve : ($3D^2$ + D – 14)y = $13e^{2x}$

(c) Solve $\left(D^2-5D+6\right)y=2\cos 3x$

MODEL QUESTION PAPER – 2
ENGINEERING MATHEMATICS – IV

Time : 3 Hrs **Max Marks : 75**

PART – A

(Marks: 15 x 1 = 15)

Answer any fifteen (15) questions

1. Find the conjugate of (1 + i) (1 – 2i).
2. If $z_1 = 2 + i$, $z_2 = 3 - 2i$ the find z_1/z_2
3. Find the quadratic equation whose root is 3 -2i.
4. Find the distance between the complex numbers 2 - I and 5 – 2i
5. State De Moivre's theorem.
6. Simplify $\frac{\cos 5\theta + i\sin 5\theta}{\cos 4\theta - i\sin 4\theta}$
7. If ω is a cube root of unity, find the value of $\omega^4 + \omega^5 + \omega^6$.
8. Solve $x^2 + 16 = 0$
9. If the mean of Poisson distribution is 1 state its probability distribution.
10. How many values does the Poisson variable take?
11. If Z is the standard normal variable find the value of $\int_{-\infty}^{\infty} f(z)dz$
12. State the normal equations to fit the straight line y=mx+c
13. Find the area bounded by the curve $y = x^3$ and x axis between x = 0 and x = 1.
14. Write the order and degree of the differential equation $y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$
15. Find the solution of $\frac{dy}{dx} = 2xy$

16. Find the integral factor of $\frac{dy}{dx}+\frac{2xy}{1+x^2}=1+x^3$

17. Solve $(D^2 +9)\,y = 0$

18. Find the particular integral of $(D^2 - 3D + 2)y = e^{-X}$

19. Find the complimentary function of $(D^2-5D+6)y=e^x$

20. Find the particular integral of $(D^2+25)y=\cos x$

Part - B

(Marks : 5 x 12 = 60)

[N.B :- (1) Answer all questions choosing any two divisions from each question.

(2) All questions carry equal marks.]

21 (a) Find the real and imaginary parts of conjugate of the complex number $\frac{(1+i)(2-i)}{(2+i)^2}$

(b) Find the modulus and amplitude of the complex number $\sqrt{3}-i$

(c) Show that the complex numbers (9 + i), (4 + 13i), (-8 + 8i), (-3 - 4i) form a Square.

22 (a) Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^4\ (\cos 4\theta + i\sin 4\theta)^{-3}}{(\cos 3\theta + i\sin 3\theta)^2\ (\cos 5\theta - \sin 5\theta)^{-2}}$

(b) If a = cos2α + sin2α, b = cos2β + sin2β, c = cos2γ + sin2γ , prove that

(i) $\sqrt{abc}+\frac{1}{\sqrt{abc}}=2\cos(\alpha+\beta+\gamma)$ *(ii)* $\frac{a^2b^2+c^2}{abc}=$ 2cos2(α+β –γ)

(c) Solve $x^5+1=0$

23 (a) If 3% of electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs exactly 5 are defective.

(b) The mean score of 1000 students in an examination is 36 and standard deviation is 16.If the score of the students is normally distributed how many students are expected to score more than 60 marks.

(c) Using the method of least squares fit the straight line

X	0	1	2	3	4
Y	1	1	3	4	6

24 (a) Find the volume of a sphere of radius *r* by Integration.

(b) Solve $\left(1-e^{x}\right)\sec^{2} y dy + 3e^{x} \tan y dx = 0$

(c) Solve $\left(1+x^{2}\right)\frac{dy}{dx} + y = 1$

25 (a) Solve : $(D^2 + D + 1)y = 0$

(b) Solve : $(D^2 - 13D + 12)y = 2e^{-2x} + 5$

(c) Solve : $(D^2 + 16)y = \sin 9x$