Lecture Notes on Turbomachinery

ME251. Instructor: R. Betti

1. Euler Equation

Consider a control volume (Euler wheel) rotating at the angular velocity $\omega = constant$ (Fig. 1). The fluid particle with mass Δm enters the c.v. with velocity $\vec{C}_1$ and it is carried by the rotating wheel. The particle leaves the c.v. with a velocity $\vec{C}_2$. Newton's law applied to the particle reads as

$$\Delta m \frac{d\vec{C}}{dt} = \vec{F} \tag{1}$$

where $\vec{F}$ is the force applied on the particle by the wheel, $\vec{c} = d\vec{x}/dt$ is the particle velocity and $\vec{x}(t)$ is the particle trajectory. From Eq. (1) it follows that

$$\Delta m \frac{d}{dt}(\vec{C} \times \vec{x}) = \vec{F} \times \vec{x} \tag{2}$$

The axial (z-)component of Eq. (2) is

$$\Delta m \frac{d}{dt}(\vec{C} \times \vec{x} \cdot \mathbf{e}_z) = \vec{F} \times \vec{x} \cdot \mathbf{e}_z \tag{3}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ are the unit vectors in the r, θ, z directions. Since $\vec{x} = r\mathbf{e}_r + z\mathbf{e}_z$, Eq. (3) yields

$$\frac{d}{dt}(\Delta m C_\theta r) = F_\theta r \tag{4}$$

Multiplying both sides of Eq. (4) by ω, remembering that $C_\theta = r\dot{\theta} = r\omega$ and integrating in time between the particle inlet and outlet leads to the following equation for the work done by the wheel on the particle

$$[\Delta m C_\theta r \omega]_1^2 = \int_1^2 F_\theta C_\theta dt = \Delta W_{12} \tag{5}$$

Obviously the work done by the particle on the wheel is just the opposite ($-\Delta W_{12}$). For the unit of mass ($\Delta m = 1$),

$$-W_{12} = \omega[C_{\theta 1} r_1 - C_{\theta 2} r_2] \tag{6}$$

Eq. (6) relates the work done by the unit of mass on the wheel with the wheel angular frequency ω, the outlet and inlet fluid tangential velocities, and the outlet and inlet fluid radial position. The power produced/absorbed by the wheel is

$$P = \dot{m}(-W_{12})$$

If $P > 0$ the wheel is a turbine; if $P < 0$, the wheel is a compressor or pump.

2. Application of the Euler Equation

Consider air flowing inside the control volume without swirl ($C_{\theta 1} = 0$) and leaving the control volume at the radial position $r_2 = 15cm$ with a tangential velocity equal to 90% of the rotor velocity. The air inlet temperature is 15 C and its pressure is 100kPa. The rotor frequency is 20,000 revolutions/minute. Determine the compression ratio P_{out}/P_{in} assuming that a diffuser is present at the outlet and the velocity at the diffuser exit is negligible. Also neglect the inlet kinetic energy with respect to the air enthalpy and the heat transfer in the compressor. Assume that there are no sources of irreversibilities.

Using Euler equation, the work done on the unit of mass of air is $W_{12} = C_{\theta 2} r_2 \omega$. Since $C_{\theta 2} = 0.9 \omega r_2$, then $W_{12} = 0.9(\omega r_2)^2$. Using $\omega = 2\pi f = 2\pi \times 20000/60 s^{-1}$, $r_2 = 0.15$m, we find $W_{12} = 88.736 kJ/kg$. The energy balance between the inlet and the outlet of the c.v. is

$$h_1 + \frac{C_1^2}{2} + W_{12} = h_2 + \frac{C_2^2}{2} \tag{7}$$

The energy balance between the inlet and the outlet of the diffuser is $h_2 + \frac{C_2^2}{2} = h_{out}$. Neglecting the kinetic energy at the inlet, Eq. (7) becomes

$$h_{in} + W_{12} = h_{out} \tag{8a}$$

From the properties of perfect gases, $dh = C_p dT$ and

$$h_{out} - h_{in} = C_p(T_{out} - T_{in}) \tag{8b}$$

Also $dh = C_p dT = vdp$ and $pv = RT$. Thus,

$$\frac{P_{out}}{P_{in}} = \left(\frac{T_{out}}{T_{in}}\right)^{\frac{C_p}{R}} \tag{9}$$

Determining T_{out}/T_{in} from Eq. (8) and substituting into Eq. (9) yields

$$\frac{P_{out}}{P_{in}} = \left[1 + \frac{W_{12}}{C_p T_{in}}\right]^{\frac{C_p}{R}} \tag{10}$$

Using $C_p = 1kJ/kg \cdot K$, $T_1 = 288K$ gives

$$\frac{P_{out}}{P_{in}} = 2.55$$

The device studied is here is a compressor with compression ratio $\beta = 2.55$. It seems that the only requirement for building such compressor is to design the wheel in such a way to induce an outlet tangential velocity equal to 90% of the rotor speed and to design the diffuser to convert kinetic energy into pressure.

How can we do that?????????

3.1 Blades in an Axial Compressors

We start by using the following shape of the control volume interior. In the control volume there are radial blades as described in Fig. 2. The air flow enters the c.v. in the axial direction (no swirl), the mass flow rate is $\dot{m} = 0.9$ kg/s, the blades radial length is $h = 1$cm. The blades are installed on a cylindrical shaft of radius $r_s = 14.5$ cm. The average radius of the blades is $r_b = 15$cm. The flow area is $A_1 = \pi[(r_s + h)^2 - r_s^2] = 94cm^2$ and the air density is $\rho_1 = p_1/RT_1 = 1.2kg/m^3$. The average flow velocity of the air entering the blades is $C_1 = \dot{m}/\rho_1 A_1 = 80$m/s and the blade velocity is $C_b = \omega r_b = 314m/s$.

In the frame of reference of the rotating blades, the air velocity is $\vec{C}_1 - \vec{C}_b = \vec{C}_{r1}$. Defining with α the deflection angle of C_{r1}, the blade must be oriented with the same angle in order to be aligned with the flow. Thus,

$$\alpha = \tan^{-1} \frac{C_1}{C_b} \tag{11}$$

Since $C_b = \omega r_b = 314 m/s$, Eq. (11) yields $\alpha = 14^o$.

In the blades frame of reference, the air leaves the blades with a velocity $\vec{C}_{r2}$ oriented like the blades. The air velocity in the rest frame of reference is $\vec{C}_2 = \vec{C}_{r2} + \vec{C}_b$. Defining with ϕ, the outlet deflection angle of the blade, it follows that

$$C_{\theta 2} = C_b - C_{r2} \cos\phi \tag{12}$$

where $C_{\theta 2}$ is the tangential component of the velocity at the exit of the rotor. In order to design the compressor described in Sec. 2, it is required that $C_{\theta 2} = 0.9 C_b$ thus leading to $C_{r2} \cos\phi = 0.1 C_b$. There are two unknowns in this equation: C_{r2} and ϕ. Now remember that according to the mass conservation, the axial flow of air is constant. That is

$$\rho_1 C_1 = \rho_2 C_{r2} \sin\phi \qquad \rightarrow \qquad C_1 = \frac{\rho_2}{\rho_1} C_{r2} \sin\phi \tag{13}$$

Furthermore, energy conservation in the blade frame of reference requires

$$h_1 + \frac{C_{r1}^2}{2} = h_2 + \frac{C_{r2}^2}{2} \tag{14}$$

Using the perfect gas model ($dh = C_p dT$, $C_p = constant$), Eq. (14) can be rewritten as

$$\frac{T_2}{T_1} = 1 + \frac{C_{r1}^2 - C_{r2}^2}{2 C_p T_1} \tag{15}$$

Neglecting heat exchanges and generation of irreversibilities, the pressure and temperature are related by the following equation

$$\frac{P_1}{P_2} = \left(\frac{T_1}{T_2}\right)^{C_p/R} \tag{16a}$$

and the densities ($\rho = P/RT$) obey

$$\frac{\rho_1}{\rho_2} = \left(\frac{T_1}{T_2}\right)^{\frac{C_p}{R}-1} = \left(\frac{T_1}{T_2}\right)^{\frac{C_v}{R}} \tag{16b}$$

Substituting Eq. (16) and (15) into Eq. (13) yields

$$C_1 = \left(1 + \frac{C_{r1}^2 - C_{r2}^2}{2 C_p T_1}\right)^{C_v/R} C_{r2} \sin\phi \tag{17}$$

Since $C_{r1} = \sqrt{C_1^2 + C_b^2} = 324m/s$ and $C_{r2} = 0.1C_b/\cos\phi$, Eq. (17) can be rewritten in the following form

$$C_1 = \left(1 + \frac{C_1^2 + C_b^2 - \frac{(0.1C_b)^2}{\cos^2\phi}}{2C_pT_1}\right)^{C_v/R} (0.1C_b)\tan\phi \tag{18}$$

Eq. (18) is the equation for the blade deflection angle ϕ. Substituting $C_1 = 80m/s$, $C_b = 314m/s$, $C_p = 1000J/kgK$ and $T_1 = 288$ K, $C_v/R = 2.48$, Eq. (18) is numerically solved and yields $\phi = 59^o$. Then using $C_{r2} = 0.1C_b/\cos\phi$ one finds $C_{r2} = 61m/s$.

3.2 The Diffuser

Observe that at the exit of the rotor blades, the air velocity is large,

$$C_2 = \sqrt{C_{r2}^2 - 2C_{r2}C_b\cos\phi + C_b^2} = 287m/s \tag{19}$$

and so is the kinetic energy

$$E_k = \frac{C_2^2}{2} = 41kJ/kg \tag{20}$$

The compression ratio between the inlet and outlet of the rotor is low. From Eq. (15) and (16a), $P_2/P_1 = 1.76$. In order to transform the kinetic energy into pressure, one can use a set of diffusers in parallel by appropriately installing a set of static blades (stator blades). The deflection angle of the air velocity at the exit of the rotor blades can be obtained by the sine rule $C_2/\sin\phi = C_{r2}/\sin\delta$ yielding

$$\delta = \sin^{-1}\left(\frac{C_{r2}}{C_2}\sin\phi\right) = 11^o \tag{20}$$

In order to follow the flow, the stator blades are also deflected by 11^o. It is usually desirable to have axial flow at the exit of the stator blades ($\vec{C}_3 = C_3\mathbf{e}_z$). Defining with A_2 and A_3 the flow areas at the inlet and outlet of the stator blades, the conservation of mass gives

$$\rho_2A_2C_2\sin\delta = \rho_3C_3A_3 \qquad \rightarrow \frac{C_2A_2}{C_3A_3} = \frac{\rho_3}{\rho_2}\frac{1}{\sin\delta} \tag{21}$$

The energy conservation in the stator reads as

$$h_2 + \frac{C_2^2}{2} = h_3 + \frac{C_3^2}{2} \tag{22}$$

and using the perfect gas model

$$\frac{T_3}{T_2} = 1 + \frac{C_2^2 - C_3^2}{2C_pT_2} \qquad \frac{P_3}{P_2} = \left(\frac{T_3}{T_2}\right)^{C_p/R} \tag{23}$$

Combining Eq. (22) and Eq. (23) yields the following equation

$$\frac{A_2}{A_3} = \frac{C_3}{C_2}\frac{1}{\sin\delta}\left[1 + \frac{C_2^2 - C_3^2}{2C_pT_2}\right]^{C_v/R} \tag{24}$$

Using $C_{r1} = 324m/s$ into Eq. (15) gives $T_2 = 338$ K and Eq. (24) becomes

$$\frac{A_2}{A_3} = 5.5\frac{C_3}{C_2}\left[1.12 - 0.12\left(\frac{C_3}{C_2}\right)^2\right]^{2.5} \tag{25}$$

For a given value of C_3/C_2, Eq. (25) yields the stator inlet/outlet ratio. For example:

$$C_3 = 20m/s \qquad C_3/C_2 = 0.07 \qquad A_2/A_3 = 0.5$$

$$C_3 = 10m/s \qquad C_3/C_2 = 0.035 \qquad A_2/A_3 = 0.2$$

For $C_3 = 20m/s$, Eq. (23) yields $T_3/T_2 = 1.12$ and $P_3/P_2 = 1.48$ leading to a compression ratio

$$\frac{P_3}{P_1} = 1.48 \times 1.76 = 2.6$$

as expected from Sec. 2.

4.1. Axial Flow Turbines

Remember the expression of the work done by the fluid on the Euler wheel (Eq. (6)). If you can design the wheel in such a way that $C_{\theta 2} = 0$ and $C_{\theta 1} > 0$, then the work $-W_{12} = \omega C_{\theta 1} r_1 > 0$ can be extracted from the fluid. Here C_θ is the velocity component in the tangential direction or direction of rotation of the wheel. Thus, it is important to induce a tangential component to the fluid entering the wheel. This can be achieved through a set of stator blades. Fig. 5 shows a stator blade deflecting the fluid by 70^o ($\alpha_1 = 20^o$). Fig. 4 shows the cross section of a turbine. Observe that the stator blades are connected to the casing and do not move. Their length changes in the axial direction (See Fig. 6. where $R_0 < R < R_1$). The Euler wheel of Fig. 1 is represented by the rotor blades that are connected to the shaft and rotate with its angular frequency. At the rotor inlet, the fluid has a tangential component $C_{\theta 1}$. At the outlet of the rotor blades of Fig. 5. the fluid velocity is deflected by the angle $\alpha_2 \simeq 90^o$ and $C_{\theta 2} = 0$.

The stator and rotor represent one stage of the turbine. There are several stages in each turbine (three stages in Fig. 4). The rotor blades have constant length in the radial direction ($R_2 = R_1$).

4.2 Example

Consider a flow of saturated steam at 70 bars and $h_{in} = h_0 = 2772.1$ kJ/kg entering the turbine with velocity $C_0 = 0.25 C_s$ where $C_s = \sqrt{5p/3\rho}$ is the speed of sound

$$C_0 = 140 m/s$$

The shaft radius is $R_s = 18$cm and the mass flow rate is $\dot{m} = 1200 kg/s$. The flow area is

$$A_0 = \frac{\dot{m} v_0}{C_0} = 0.23 m^2 \tag{26}$$

corresponding to a inlet radius $R_0 = 32$cm. The steam pressure drops from 70 to 10 bars at constant entropy. The quality at the exit of the turbine is

$$x_{out} = \frac{5.8133 - 2.1387}{6.5863 - 2.1387} = 0.826 \tag{27}$$

The enthalpy at the exit of the turbine is

$$h_{out} = 0.826 \times 2778.1 + (1 - 0.826) \times 762.81 = 2427 kJ/kg \tag{28}$$

In our design, the enthalpy drop is the same through each stage and the turbine has three stages (Fig. 4)

$$h_0 - h_2 = \frac{h_{in} - h_{out}}{3} = 115 kg/kJ \tag{29}$$

The stage reaction is defined as the ratio of the enthalpy drop through the rotor and the total enthalpy drop through the stage

$$Reaction = \frac{h_1 - h_2}{h_0 - h_2} \tag{30}$$

For a 50% reaction stage, $h_0 - h_1 = 57.5 kJ/kg$ and $h_1 - h_2 = 57.5 kJ/kg$.

4.3 Stator Blades

The energy conservation through the stator yields

$$h_0 + \frac{C_0^2}{2} = h_1 + \frac{C_1^2}{2} \tag{31}$$

The mass conservation yields

$$\rho_0 C_0 A_0 = \rho_1 C_1 A_1 \sin\alpha_1 \tag{32}$$

From the energy conservation $C_1 = 367$m/s. If we knew the pressure p_1, we could calculate the density ρ_1 to use in the mass conservation equation. The equation for p_1 is obtained by

$$x_1 = \frac{s_1 - s_l(p_1)}{s_v(p_1) - s_l(p_1)} = \frac{h_1 - h_l(p_1)}{h_v(p_1) - h_l(p_1)} \tag{33}$$

where $s_1 = s_0$ is the entropy of the mixture, s_l, s_v, h_l, h_v are the entropies and enthalpies of the saturated liquid and vapor at p_1 Obviously Eq. (33) has one unknown: the pressure p_1. I try a few values of p_1 and I found that $p_1 = 52$ bars is close to the exact solution. This value of p_1 leads to $x_1 = 0.95$ and

$$v_1 = x_1 v_v(p_1) + (1 - x_1) v_l(p_1) = 0.036 m^3/kg \tag{34}$$

In order to maximize $C_{\theta 1}$, we choose a small deflection angle $\alpha_1 = 20^o$ thus leading to

$$A_1 = A_0 \frac{v_1 C_0}{v_0 C_1 \sin \alpha_1} = 1.48 \times 0.23 = 0.34 m^2 \tag{35}$$

corresponding to $R_1 = 38$cm.

4.4 Rotor Blades

The rotor blades are connected to the shaft that rotates at 6000rpm. The average radius of the rotor blades (Fig. 6) is

$$R_b = \frac{R_s + R_1}{2} = 28cm \tag{36}$$

The blade velocity at the midpoint is $C_b = \omega R_b = 176 m/s$. In the frame of reference of the rotor (Fig. 5), the fluid velocity is

$$C_{r1} = \sqrt{C_1^2 + C_b^2 - 2C_b C_1 \cos \alpha_1} = 210 m/s \tag{37}$$

The angle β_1 is

$$\beta_1 = \sin^{-1}\left(\frac{C_1}{C_{r1}} \sin \alpha_1\right) = 37^o \tag{38}$$

In the rotor frame of reference, the conservation of energy reads as

$$h_1 + \frac{C_{r1}^2}{2} = h_2 + \frac{C_{r2}^2}{2} \tag{39}$$

This equation can be solved to find C_{r2}

$$C_{r2} = \sqrt{2(h_1 - h_2) + C_{r1}^2} = 399 m/s$$

The angle β_2 can be determined by the conservation of the axial flow

$$\rho_2 C_{r2} \sin \beta_2 A_2 = \rho_1 C_{r1} \sin \beta_1 A_1 \tag{40}$$

where $A_2 = A_1$ (Fig. 6). To determine ρ_2 we need the quality and the pressure at 2

$$\rho_2^{-1} = v_2 = x_2 v_v(p_2) + (1 - x_2) v_l(p_2) \tag{41}$$

For isentropic flows, the pressure p_2 can be determined from the quality equation

$$x_2 = \frac{s_2 - s_l(p_2)}{s_v(p_2) - s_l(p_2)} = \frac{h_2 - h_l(p_2)}{h_v(p_2) - h_l(p_2)} \tag{42}$$

where $s_2 = s_1 = s_0 = 5.8133$ and $h_2 = h_0 - 115 = 2657 kJ/kg$. I tried a few values of p_2 and $p_2 = 40$ bars seems to be close enough to the exact solution. The corresponding quality is $x_2 = 0.92$ and the specific volume is $v_2 = 0.046$. Eq. (40) can be solved to find β_2

$$\beta_2 = \sin^{-1}\left(\frac{v_2 C_{r1}}{v_1 C_{r2}} \sin\beta_1\right) = 24^o \tag{43}$$

The velocity of the fluid in the lab. frame of reference is

$$C_2 = \sqrt{C_{r2}^2 + C_b^2 - 2C_{r2}C_b\cos\beta_2} = 248 m/s \tag{44}$$

and $\alpha_2 = 40^o$. Observe that the angle α_2 is not 90^o as we wished. We could change the degree of reaction until we find $\alpha_2 = 90^o$. However, observe that values of $\alpha_2 < 90$ make $C_{\theta 2} < 0$ and therefore they increase the power transfer from the fluid to the rotor, thus improving the performance of the turbine. The energy variation of the fluid is

$$\Delta E = h_0 + \frac{C_0^2}{2} - h_2 - \frac{C_2^2}{2} = 94 kJ/kg$$

This is the work transferred from the fluid to the turbine.