

**Time : 3 hours]**

**Note :** (i) Attempt All the problems. The choice of problems is internal as indicated.

**Q. 1. Attempt any Four parts of the following : —** 5 × 4 = 20

(a) A 4 kg object falls from rest at time  $t = 0$  in a medium offering a resistance in kg numerically equal to twice its instantaneous velocity in m/sec. Find the velocity and distance travelled at any time  $t > 0$ , and also the limiting velocity.

**Ans.** Let  $v$  m/sec be the velocity of the object.

Accelerating force =  $(4 - 2v)$  N

The equation of motion is

$$\frac{4}{g} \frac{dv}{dt} = 4 - 2v$$

$$\frac{2dv}{4 - 2v} = \frac{g}{2} dt$$

$$-\log(4 - 2v) = \frac{gt}{2} + c_1 \quad \dots(1)$$

when  $t = 0, v = 0 \Rightarrow c_1 = -\log 4$

$$(1) \Rightarrow -\log(4 - 2v) = \frac{gt}{2} - \log 4$$

$$\log\left(\frac{4}{4 - 2v}\right) = \frac{gt}{2}$$

$$\frac{4}{4 - 2v} = e^{gt/2}$$

$$\frac{4 - 2v}{4} = e^{-gt/2}$$

$$4 - 2v = 4e^{-gt/2}$$

$$v = 2(1 - e^{-gt/2}) \quad \dots(2)$$

which gives the velocity of the object.

The limiting velocity is the velocity when  $t \rightarrow \infty$  from (2), the limiting velocity = 2

**[Total Marks : 100**

Now from (2)  $\rightarrow$

$$\frac{dx}{dt} = 2(1 - e^{-gt/2})$$

$$dx = 2(1 - e^{-gt/2})dt$$

$$\text{Integrating, } x = 2\left(t + \frac{2}{g}e^{-gt/2}\right) + c_2 \quad \dots(3)$$

Initially when  $t = 0, x = 0$

$$c_2 = -\frac{4}{g}$$

$$(3) \Rightarrow x = 2\left(t + \frac{2}{g}e^{-gt/2}\right) - \frac{4}{g}$$

$$x = 2t - \frac{4}{g}(1 - e^{-gt/2})$$

**Q. 1. (b) Solve**

$$(3y - 2xy^3)dx + (4x - 3x^2y^2)dy = 0$$

$$\text{Ans. } (3y - 2xy^3)dx + (4x - 3x^2y^2)dy = 0 \quad \dots(1)$$

Multiplying by  $x^h y^k$ , we have

$$(3x^h y^{k+1} - 2x^{h+1} y^{k+3})dx +$$

$$(4x^{h+1} y^k - 3x^{h+2} y^{k+2})dy = 0$$

For this equation to be exact we must

have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$3(k+1)x^h y^k - 2(k+3)x^{h+1} y^{k+2}$$

$$= 4(h+1)x^h y^k - 3(h+2)x^{h+1} y^{k+2}$$

which holds when

$$3(k+1) = 4(h+1) \text{ and } -2(k+3) = -3(h+2)$$

$$= -3(h+2)$$

solving these equations, we have

$$h = 2, k = 3$$

$$\text{I.F.} = x^2 y^3$$

Multiplying (1) by  $x^2 y^3$ , we have

$$(3x^2 y^4 - 2x^3 y^6)dx$$

which is exact  
The solution is

$$\int (3x^2y^4 - 2x^3y^6)dx = c$$

$y$  constant

$$x^3y^4 - \frac{x^4}{2}y^6 = c$$

Q. 1. (c) solve

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = e^x$$

Ans.  $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 2y = e^x$

$$(D^3 - D^2 - 4D - 2)y = e^x$$

A.E is  $D^3 - D^2 - 4D - 2 = 0$

$$(D+1)(D^2 - 2D - 2) = 0$$

$$D = -1, \frac{2 \pm \sqrt{4+8}}{2}$$

$$D = -1, 1 \pm i\sqrt{3}$$

$$C.F. = c_1e^{-x} + e^x(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3})$$

$$P.I. = \frac{1}{D^3 - D^2 - 4D - 2} e^x$$

$$= \frac{1}{1-1-4-2} e^x$$

$$= -\frac{e^x}{6}$$

$$y = C.F. + P.I.$$

$$y = c_1e^{-x} + e^x(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) - \frac{e^x}{6}$$

Q. 1. (d) solve  $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 4 \cot 2x$

Ans.  $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = 4 \cot 2x$

A.E. is  $D^3 + 4D = 0$

$$D(D^2 + 4) = 0$$

$$D = 0, \pm 2i$$

$$C.F. = c_1e^{0x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$$

$$= c_1 + c_2 \cos 2x + c_3 \sin 2x$$

$$P.I. = \frac{1}{D^3 + 4D} 4 \cot 2x$$

$$\frac{1}{D^3 + 4D} 4 \cot 2x$$

$$= \frac{1}{D^2 + 4} \frac{4}{2} \log \sin 2x$$

$$= 2 \frac{1}{D^2 + 4} \log \sin 2x$$

$$= \frac{2}{4i} \left[ \frac{1}{D-2i} - \frac{1}{D+2i} \right] \log \sin 2x$$

$$= \frac{1}{2i} \left[ \frac{1}{D-2i} - \frac{1}{D+2i} \right] \log \sin 2x$$

$$= \frac{1}{2i} [P_1 - P_2]$$

...(1)

$$P_1 = \frac{1}{D-2i} \log \sin 2x$$

$$= e^{2ix} \int e^{-2ix} \log \sin 2x dx$$

$$= e^{2ix} \left[ \log \sin 2x \left( \frac{-e^{-2ix}}{2i} \right) - \int \frac{2 \cos 2x}{\sin 2x} \right.$$

$$\left. \left( \frac{-e^{-2ix}}{2i} \right) dx \right]$$

$$= -\frac{1}{2i} \log \sin 2x + \frac{2e^{2ix}}{2i}$$

$$\int \frac{\cos 2x}{\sin 2x} (\cos 2x - i \sin 2x) dx$$

$$= -\frac{1}{2i} \log \sin 2x + \frac{2e^{2ix}}{2i}$$

$$\int \left( \frac{\cos^2 2x}{\sin 2x} - i \cos 2x \right) dx$$

$$= -\frac{1}{2i} \log \sin 2x + \frac{2e^{2ix}}{2i}$$

$$\int (\cos \sec 2x - \sin 2x - i \cos 2x) dx$$

$$= -\frac{1}{2i} \log \sin 2x + \frac{2e^{2ix}}{2i}$$

$$\left[ \frac{1}{2} \log (\cos \sec 2x - \cot 2x) + \frac{\cos 2x}{2} \right.$$

$$\left. - i \frac{\sin 2x}{2} \right]$$

$$= -\frac{1}{2i} \log \sin 2x + \frac{e^{2ix}}{2i}$$

$$\left[ \log (\cos \sec 2x - \cot 2x) + \cos 2x - i \sin 2x \right]$$

...(2)

$$\begin{aligned}
 P_2 &= \frac{1}{D+2i} \log \sin 2x \\
 &= e^{-2ix} \int e^{2ix} \log \sin 2x \, dx \\
 &= e^{-2ix} \left[ \log \sin 2x \left( \frac{e^{2ix}}{2i} \right) - \int \frac{2 \cos 2x}{\sin 2x} \left( \frac{e^{2ix}}{2i} \right) dx \right] \\
 &= \frac{1}{2i} \log \sin 2x - \frac{2e^{-2ix}}{2i}
 \end{aligned}$$

$$\begin{aligned}
 &\int (\cos \operatorname{ec} 2x - \sin 2x + i \cos 2x) dx \\
 &= \frac{1}{2i} \log \sin 2x - \frac{e^{-2ix}}{2i} \\
 &[\log (\cos \operatorname{ec} 2x - \cot 2x) + \cos 2x + i \sin 2x] \quad \dots(3)
 \end{aligned}$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{2i} \left( \frac{-2}{2i} \right) \log \sin 2x \\
 &+ \frac{1}{2i} \left( \frac{e^{2ix} + e^{-2ix}}{2i} \right) \log (\cos \operatorname{ec} 2x - \cot 2x) \\
 &\quad + \frac{1}{2i} \left( \frac{e^{2ix} + e^{-2ix}}{2i} \right) \cos 2x \\
 &\quad - \frac{i}{2i} \left( \frac{e^{2ix} - e^{-2ix}}{2i} \right) \sin 2x \\
 &= \frac{1}{2} \log \sin 2x - \frac{1}{2} \cos 2x \log (\cos \operatorname{ec} 2x - \cot 2x) - \frac{1}{2} \cos^2 2x - \frac{1}{2} \sin^2 2x \\
 &= \frac{1}{2} \log \sin 2x - \frac{\cos 2x}{2} \\
 &\log (\cos \operatorname{ec} 2x - \cot 2x) - \frac{1}{2} \\
 y &= \text{C.F.} + \text{P.I.}
 \end{aligned}$$

$$\begin{aligned}
 &= c_1 + c_2 \cos 2x + c_3 \sin 2x - \\
 &\frac{1}{2} (1 - \log \sin 2x) - \frac{\cos 2x}{2} \log (\cos \operatorname{ec} 2x - \cot 2x)
 \end{aligned}$$

Q. 1. (e) solve  $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$ ,

$$\begin{aligned}
 \text{Ans. } \frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x &= e^{-t}, \\
 \frac{d^2y}{dt} - 4 \frac{dx}{dt} + 3y &= \sin 2t \\
 \Rightarrow (D^2 + 3)x + Dy &= e^t \quad \dots(1)
 \end{aligned}$$

$$-4Dx + (D^2 + 3)y = \sin 2t \quad \dots(2)$$

Eliminating  $y$  from (1) & (2), we get  $[(D^2 + 3)(D^2 + 3) + 4D^2]x$

$$\begin{aligned}
 &= (D^2 + 3)e^t - D \sin 2t \\
 &(D^4 + 6D^2 + 9 + 4D^2)x \\
 &= e^t + 3e^t - 2 \cos 2t \\
 &(D^4 + 10D^2 + 9)x = 4e^t - 2 \cos 2t
 \end{aligned}$$

$$\text{A.E. is } D^4 + 10D^2 + 9 = 0$$

$$(D^2 + 9)(D^2 + 1) = 0$$

$$D = \pm 3i, \pm i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^4 + 10D^2 + 9} (4e^t - 2 \cos 2t) \\
 &= \frac{1}{D^4 + 10D^2 + 9} 4e^t - \frac{1}{(D^2 + 9)(D^2 + 1)} 2 \cos 2t \\
 &= 4 \frac{1}{1 + 10 + 9} e^t - 2 \frac{1}{(-4 + 9)(-4 + 1)} \cos 2t
 \end{aligned}$$

$$= \frac{4}{20} e^t + \frac{2}{15} \cos 2t$$

$$= \frac{e^t}{5} + \frac{2}{15} \cos 2t$$

$$\begin{aligned}
 x &= c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t \\
 &\quad + \frac{e^t}{5} + \frac{2}{15} \cos 2t \quad \dots(3)
 \end{aligned}$$

$$\frac{dx}{dt} = -c_1 \sin t + c_2 \cos t - 3c_3 \sin 3t$$

$$+ 3c_4 \cos 3t + \frac{e^t}{5} - \frac{4}{15} \sin 2t$$

$$\frac{d^2x}{dt^2} = -c_1 \cos t - c_2 \sin t - 9c_3 \cos 3t$$

$$- 9c_4 \sin 3t + \frac{e^t}{5} - \frac{8}{15} \cos 2t$$

$$\begin{aligned} \frac{dy}{dt} &= e^t - \frac{d^2x}{dt^2} - 3x \\ &= e^t + c_1 \cos t + c_2 \sin t + 9c_3 \cos 3t + \\ &9c_4 \sin 3t - \frac{e^t}{5} + \frac{8}{15} \cos 2t - 3c_1 \cos t \\ &- 3c_2 \sin t - 3c_3 \cos 3t - 3c_4 \sin 3t - \frac{3e^t}{5} \\ &\quad - \frac{6}{15} \cos 2t \\ \frac{dy}{dt} &= -2c_1 \cos t - 2c_2 \sin t + 6c_3 \cos 3t \\ &\quad + 6c_4 \sin 3t - \frac{e^t}{5} + \frac{2}{15} \cos 2t \\ y &= -2c_1 \sin t + 2c_2 \cos t + 6c_3 \frac{\sin 3t}{3} \\ &\quad - 6c_4 \frac{\cos 3t}{3} - \frac{e^t}{5} + \frac{2}{15} \frac{\sin 2t}{2} \\ y &= -2c_1 \sin t + 2c_2 \cos t + 2c_3 \sin 3t \\ &\quad - 2c_4 \cos 3t - \frac{e^t}{5} + \frac{\sin 2t}{15} \dots (4) \end{aligned}$$

Equations (3) & (4) give the required solutions

Q. 1. (f) Solve  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - \lambda^2 y = 0$

Ans.  $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - \lambda^2 y = 0$

Put  $x = e^z, z = \log x, D = \frac{d}{dz}$

$[D(D-1) + D - \lambda^2]y = 0$

$(D^2 - \lambda^2)y = 0$

A.E. is  $D^2 - \lambda^2 = 0$

$D^2 = \lambda^2 \Rightarrow D = \pm \lambda$

$y = C.F. = c_1 e^{\lambda z} + c_2 e^{-\lambda z} = c_1 x^\lambda + c_2 x^{-\lambda}$

Q. 2. Attempt any Four parts of the following : — 5 × 4 = 20

(a) Find the Laplace transform of

$$f(t) = \begin{cases} t^2 & 0 < t < 2 \\ 7 & t > 2 \end{cases}$$

Ans.  $f(t) = \begin{cases} t^2 & 0 < t < 2 \\ t-1 & 2 < t < 3 \\ 7 & t > 3 \end{cases}$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$$\begin{aligned} \bar{f}(s) &= \int_0^2 e^{-st} t^2 dt + \int_2^3 e^{-st} (t-1) dt \\ &\quad + \int_3^\infty e^{-st} 7 dt \end{aligned}$$

$$= \left[ t^2 - \left( \frac{e^{-st}}{s} \right) - (2t) \left( + \frac{e^{-st}}{s^2} \right) + (2) \right]$$

$$\left[ -\frac{e^{-st}}{s^3} \right]_0^2 + \left[ (t-1) \left( -\frac{e^{-st}}{s} \right) - \left( \frac{e^{-st}}{s^2} \right) \right]_2^3 + 7 \left[ -\frac{e^{-st}}{s} \right]_3^\infty$$

$$\begin{aligned} &= \left( -\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3} \right) \\ &\quad + \left( \frac{2}{s} e^{-3s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} \right) \\ &\quad + 7 \left( 0 + \frac{e^{-3s}}{s} \right) \end{aligned}$$

$$\begin{aligned} &= \left( -\frac{4}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{1}{s} + \frac{1}{s^2} \right) e^{-2s} \\ &\quad + \left( -\frac{2}{s} - \frac{1}{s^2} + \frac{7}{s} \right) e^{-3s} + \frac{2}{s^3} \end{aligned}$$

$$\begin{aligned} &= \left( -\frac{3}{s} - \frac{3}{s^2} - \frac{2}{s^3} \right) e^{-2s} + \left( \frac{5}{s} - \frac{1}{s^2} \right) e^{-3s} \\ &\quad + \frac{2}{s^3} \end{aligned}$$

$$\begin{aligned} \bar{f}(s) &= \frac{2}{s^3} - \left( \frac{3}{s} + \frac{3}{s^2} + \frac{2}{s^3} \right) e^{-2s} \\ &\quad + \left( \frac{5}{s} - \frac{1}{s^2} \right) e^{-3s} \end{aligned}$$

Q. 2. (b) If  $L\{f(t)\} = F(s)$ , show that

$L\left\{ \frac{1}{t} f(t) \right\} = \int_s^\infty F(s) ds$  Hence, find the

$$f(t) = \int_0^t \frac{\sin \tau}{\tau} d\tau$$

$$\text{Ans. } L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

Integrating both sides w.r. to s from 0 to  $\infty$

$$\int_s^\infty F(s) ds = \int_s^\infty \left( \int_0^\infty e^{-st} f(t) dt \right) ds$$

$$= \int_0^\infty \int_s^\infty f(t) e^{-st} ds dt$$

on changing the order of integration

$$= \int_0^\infty f(t) \left( \int_s^\infty e^{-st} ds \right) dt$$

$$= \int_0^\infty f(t) \left( -\frac{e^{-st}}{t} \right)_s^\infty dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left\{ \frac{f(t)}{t} \right\}$$

$$L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(s) ds$$

$$L\{\sin t\} = \frac{1}{s^2 + 1} = F(s)$$

$$L\left\{ \frac{\sin t}{t} \right\} = \int_s^\infty \frac{1}{s^2 + 1} ds = [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = F_1(s)$$

$$L\left\{ \int_0^t \frac{\sin \tau}{\tau} d\tau \right\} = \frac{F_1(s)}{s} = \frac{1}{s} \left( \frac{\pi}{2} - \tan^{-1} s \right)$$

Q. 2. (c) Find the function whose Laplace transform is  $\ln\left(1 + \frac{1}{s}\right)$

Ans. Let

$$f(t) = L^{-1}\left\{ \log\left(1 + \frac{1}{s}\right) \right\} = L^{-1}\left\{ \log\left(\frac{s+1}{s}\right) \right\}$$

$$f(t) = L^{-1}\{\log(s+1) - \log s\}$$

$$tf(t) = L^{-1}\left\{ -\frac{d}{ds} \{\log(s+1) - \log s\} \right\}$$

$$= L^{-1}\left\{ -\frac{1}{s+1} + \frac{1}{s} \right\}$$

$$tf(t) = -e^{-t} + 1$$

$$f(t) = \frac{1 - e^{-t}}{t}$$

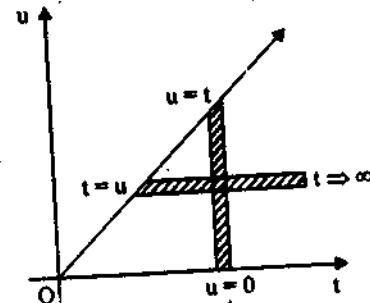
Q. 2. (d) State and prove convolution theorem for Laplace transform.

Ans. Convolution Theorem :

If  $L^{-1}\{F(s)\} = f(t)$  and

$$L^{-1}\{G(s)\} = g(t)$$

$$\text{then } L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u) du$$



$$\text{Proof: Let } \phi(t) = \int_0^t f(u)g(t-u) du$$

$$L\{\phi(t)\} = \int_0^\infty e^{-st} \left\{ \int_0^t f(u)g(t-u) du \right\} dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u) du dt \quad \dots(1)$$

On changing the order of integration, we get,

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u) dt du$$

$$= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-s(t-u)} g(t-u) dt \right\} du$$

$$= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-sv} g(v) dv \right\} du$$

putting  $t-u=v$

$$= \int_0^\infty e^{-su} f(u) G(s) du = \left( \int_0^\infty e^{-su} f(u) du \right) G(s)$$

$$= F(s) \cdot G(s)$$

$$L^{-1}\{F(s)G(s)\} = f(t) = \int_0^t f(u)g(t-u)du.$$

**Q. 2. (e) Solve for  $y(t)$  the equation**

$$y(t) = 1 + \int_0^t y(\tau) \cos(t-\tau) d\tau$$

$$\text{Ans. } y(t) = 1 + \int_0^t y(\tau) \cos(t-\tau) d\tau$$

$$y = 1 + y * \cos t$$

Taking Laplace transform,  $L(y) = Y(s)$

$$Y(s) = \frac{1}{s} + Y(s) \frac{s}{s^2+1}$$

$$Y(s) \left[ 1 - \frac{s}{s^2+1} \right] = \frac{1}{s}$$

$$Y(s) = \frac{s^2+1}{s(s^2-s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2-s+1}$$

$$A=1, B=0, C=1 \quad Y(s) = \frac{1}{s} + \frac{1}{s^2-s+1}$$

Taking inverse Laplace transform

$$y(t) = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^2-s+1}\right\}$$

$$= 1 + L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2 - \frac{3}{4}}\right\}$$

$$= 1 + e^{t/2} L^{-1}\left\{\frac{1}{s^2 - \left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= 1 + \frac{e^{t/2}}{\sqrt{3}/2} \sinh\left(\frac{t\sqrt{3}}{2}\right)$$

$$y(t) = 1 + \frac{2}{\sqrt{3}} e^{t/2} \sinh\left(\frac{t\sqrt{3}}{2}\right)$$

**Q. 2. (f) Solve using Laplace**

$$y(0) = -2, y'(0) = 8$$

$$\text{Ans. } y''(t) + 4y'(t) + 4y(t) = 6e^{-t}$$

$$y(0) = -2 \quad y'(0) = 8$$

Taking Laplace transform, we get

$$L(y'') + 4L(y') + 4L(y) = 6L(e^{-t})$$

$$s^2L(y) - sy(0) - y'(0) + 4[sL(y) - y(0)]$$

$$+ 4L(y) = \frac{6}{s+1}$$

$$L(y)(s^2 + 4s + 4) + 2s - 8 + 8 = \frac{6}{s+1}$$

$$L(y)(s+2)^2 = \frac{6}{s+1} - 2s$$

$$L(y) = \frac{6}{(s+1)(s+2)^2} - \frac{2s}{(s+2)^2}$$

$$L(y) = \frac{6}{s+1} - \frac{6}{s+2} - \frac{6}{(s+2)^2}$$

$$- \frac{2(s+2-2)}{(s+2)^2}$$

Taking inverse Laplace transform

$$y = 6L^{-1}\left\{\frac{1}{s+1}\right\} - 6L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$- 6L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 2L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2}\right\}$$

$$= 6e^{-t} - 6e^{-2t} - 6e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} - 2e^{-2t}L^{-1}$$

$$\left\{\frac{s-2}{s^2}\right\}$$

$$= 6e^{-t} - 6e^{-2t} - 6e^{-2t}t - 2e^{-2t}L^{-1}$$

$$\left\{\frac{1}{s} - \frac{2}{s^2}\right\}$$

$$= 6e^{-t} - 6e^{-2t} - 6te^{-2t} - 2e^{-2t}(1-2t)$$

$$y(t) = 6e^{-t} - 8e^{-2t} - 2te^{-2t}$$

**Q. 3. Attempt any Two parts of the following : —** 10 × 2 = 20

(a) Expand  $f(x) = k, 0 < x < 2$  in a

half range

(1) Sine series

(2) Cosine series

Ans.  $f(x) = k, 0 < x < 2$

(i) Half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$\text{where, } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{2} dx$$

$$= \frac{2}{2} \int_0^2 k \sin \frac{n\pi x}{2} dx$$

$$= k \left[ -\frac{\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{2k}{n\pi} [-\cos n\pi + 1]$$

$$= \frac{2k}{n\pi} [1 - (-1)^n]$$

$$f(x) = \frac{2k}{\pi} \sum_{n=1}^{\infty} [1 - (-1)^n] \sin \frac{n\pi x}{2}$$

(ii) Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{2}$$

where

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{2} \int_0^2 k dx = k(x)_0^2$$

$$a_0 = 2k$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{2} dx$$

$$= \frac{2}{2} \int_0^2 k \cos \frac{n\pi x}{2} dx$$

$$= k \left[ \frac{\sin \frac{n\pi x}{2}}{(\frac{n\pi}{2})} \right]_0^2 = \frac{2k}{n\pi} (\sin n\pi - \sin 0)$$

$$= 0$$

$$f(x) = \frac{2k}{2} = k$$

Q. 3. (b) If  $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ \pi(2-x) & 1 < x < 2 \end{cases}$

using half range cosine series expansion, show that

$$\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$$

$$\text{Ans. } f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$$

Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^1 \pi x dx$$

$$+ \int_1^2 \pi(2-x) dx$$

$$= \pi \left( \frac{x^2}{2} \right)_0^1 + \pi \left( 2x - \frac{x^2}{2} \right)_1^2$$

$$= \frac{\pi}{2} + \pi \left[ (4-2) - \left( 2 - \frac{1}{2} \right) \right]$$

$$= \frac{\pi}{2} + \pi \left[ 2 - \frac{3}{2} \right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx$$

$$= \int_0^1 \pi(x) \cos \frac{n\pi x}{2} dx + \int_1^2 \pi(2-x)$$

$$\cos \frac{n\pi x}{2} dx$$

$$= \pi \left[ x \left( \frac{\sin \frac{n\pi x}{2}}{n\pi/2} \right) - \left( -\frac{\cos \frac{n\pi x}{2}}{n^2 \pi^2 / 4} \right) \right]_0^1 + \pi$$

$$\left[ (2-x) \left( \frac{\sin \frac{n\pi x}{2}}{n\pi/2} \right) - (-1) \left( -\frac{\cos \frac{n\pi x}{2}}{n^2 \pi^2 / 4} \right) \right]_1^2$$

$$= \pi \left[ \left( \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2 \pi^2} \cos \frac{n\pi}{2} \right) - \left( \frac{4}{n^2 \pi^2} \right) \right]$$

$$+ \pi \left[ -\frac{4}{n^2 \pi^2} \cos n\pi - \left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n^2 \pi^2} \cos \frac{n\pi}{2} \right\} \right]$$

$$= \pi \left[ \frac{4}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2 \pi^2} - \frac{4}{n^2 \pi^2} \cos n\pi \right]$$

$$\begin{aligned}
& -\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} \\
& = \pi \left[ \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2\pi^2} \right. \\
& \quad \left. - \frac{4}{n^2\pi^2} \cos n\pi - \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} \right] \\
& = \frac{4\pi}{n^2\pi^2} \left[ 2 \cos \frac{n\pi}{2} - 1 - \cos n\pi \right] \\
& = \frac{4}{n^2\pi} \left[ 2 \cos \frac{n\pi}{2} - 1 - \cos n\pi \right]
\end{aligned}$$

when  $n$  is odd  $\cos \frac{n\pi}{2} = 0$  and  $\cos n\pi = -1$

$$a_n = 0 \Rightarrow a_1 = a_3 = a_5 \dots = 0$$

When  $n$  is even

$$a_2 = \frac{4}{2^2\pi} [2 \cos \pi - 1 - \cos 2\pi] = -\frac{16}{2^2\pi}$$

$$a_4 = \frac{4}{4^2\pi} [2 \cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{4}{6^2\pi} [2 \cos 3\pi - 1 - \cos 6\pi] = -\frac{16}{6^2\pi}$$

$$f(x) = \frac{\pi}{2} - \frac{16}{\pi} \left( \frac{1}{2^2} \cos \frac{2\pi x}{2} + \frac{1}{6^2} \cos \frac{6\pi x}{2} + \dots \right)$$

$$f(x) = \frac{\pi}{2} - \frac{16}{\pi} \left( \frac{1}{2^2} \cos \pi x + \frac{1}{6^2} \cos 3\pi x + \dots \right)$$

$$a_0^2 = \pi^2, a_2^2 = \frac{(16)^2}{2^4\pi^2}, a_4^2 = \frac{(16)^2}{6^4\pi^2} \dots$$

The root mean square value,

$$(\bar{y})^2 = \frac{1}{2} \int_0^2 y^2 dx$$

$$(\bar{y})^2 = \frac{1}{2} \left[ \int_0^1 \pi^2 x^2 dx + \int_1^2 \pi^2 (2-x)^2 dx \right]$$

$$= \frac{\pi^2}{2} \left[ \left( \frac{x^3}{3} \right)_0^1 + \int_1^2 (4+x^2-4x) dx \right]$$

$$= \frac{\pi^2}{2} \left[ \frac{1}{3} + \left( 4x + \frac{x^3}{3} - 4 \frac{x^2}{2} \right)_1^2 \right]$$

$$= \frac{\pi^2}{2} \left[ \frac{1}{3} + \left( 8 + \frac{8}{3} - 8 \right) - \left( 4 + \frac{1}{3} - 2 \right) \right]$$

$$= \frac{\pi^2}{2} \left[ \frac{1}{3} + \frac{8}{3} - 2 - \frac{1}{3} \right] = \frac{\pi^2}{2} \left( \frac{2}{3} \right)$$

$$= \frac{\pi^2}{3}$$

$$\frac{\pi^2}{3} = \frac{2}{22} \left[ \frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + a_4^2 + \dots \right]$$

$$\frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{(16)^2}{2^4\pi^2} + \frac{(16)^2}{6^4\pi^2} + \frac{(16)^2}{10^4\pi^2} \dots$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{(16)^2}{2^4\pi^2} \left( 1 + \frac{1}{3^4} + \frac{1}{5^4} \dots \right)$$

$$\frac{\pi^2}{6} = \frac{(16)^2}{2^4\pi^2} \left( 1 + \frac{1}{3^4} + \frac{1}{5^4} \dots \right)$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = 1 + \frac{1}{3^4} + \frac{1}{5^4} \dots$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \dots$$

**Q. 3. (c) Solve**

$$\frac{\partial^3 u}{\partial x^3} - 3 \frac{\partial^3 u}{\partial x^2 \partial y} + 4 \frac{\partial^3 u}{\partial y^3} = e^{x+2y}$$

$$\text{Ans. } \frac{\partial^3 u}{\partial x^3} - 3 \frac{\partial^3 u}{\partial x^2 \partial y} + 4 \frac{\partial^3 u}{\partial y^3} = e^{x+2y}$$

$$(D^3 - 3D^2 D' + 4D'^3)u = e^{x+2y}$$

$$A. E \text{ is } m^3 - 3m^2 + 4 = 0$$

$$m = -1, 2, 2$$

$$C.F. = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I. = \frac{1}{D^3 - 3D^2 D' + 4D'^3} e^{x+2y}$$

$$D = 1, D' = 2$$

$$= \frac{1}{1 - 3 \times 2 + 4 \times 8} e^{x+2y}$$

$$= \frac{1}{27} e^{x+2y}$$



$$u = C.F + P.I$$

$$= f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Q. 4. Attempt any Two parts of the following: —  $10 \times 2 = 20$

(a) Find the solution (a) using the power series method  $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + x^2 y = 0$ .

Ans.  $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + x^2 y = 0$  ... (1)

Here  $x = 0$  is a regular singular point.

Let  $y = \sum a_m x^{m+k}$  ... (2)

$$\frac{dy}{dx} = \sum a_m (m+k) x^{m+k-1}$$

$$\frac{d^2 y}{dx^2} = \sum a_m (m+k)(m+k-1) x^{m+k-2}$$

$$x \sum a_m (m+k)(m+k-1) x^{m+k-2}$$

$$+ \sum a_m (m+k) x^{m+k-1}$$

$$+ x^2 \sum a_m x^{m+k} = 0$$

$$\sum a_m [(m+k)(m+k-1) + (m+k)]$$

$$x^{m+k-1} + \sum a_m x^{m+k+2} = 0$$

$$\sum a_m (m+k)^2 x^{m+k-1}$$

$$+ \sum a_m x^{m+k+2} = 0$$
 ... (3)

Equating to zero the coefficient of  $x^{k-1}$

$$a_0 k^2 = 0 \Rightarrow k^2 = 0$$

$$k = 0, 0$$

Equating to zero the coefficient of  $x^k$

$$a_1 (k+1)^2 = 0 \Rightarrow a_1 = 0$$

Equating to zero the coefficient of  $x^{k+1}$

$$a_2 (k+2)^2 = 0 \Rightarrow a_2 = 0$$

Equating to zero the coefficient of  $x^{m+k+2}$

$$a_{m+2} = -\frac{a_m}{(m+k+3)^2}$$
 ... (4)

$$a_3 = -\frac{1}{(k+3)^2} a_0$$

$$a_4 = -\frac{1}{(k+4)^2} a_1 = 0$$

$$a_4 = a_7 = a_{10} = 0$$

$$a_5 = -\frac{1}{(k+5)^2} a_2 = 0, a_8 = 0, a_{11} = 0$$

$$a_6 = -\frac{1}{(k+6)^2} a_3 = \frac{1}{(k+3)^2 (k+6)^2} a_0$$

$$a_9 = -\frac{1}{(k+9)^2} a_6$$

$$= -\frac{1}{(k+3)^2 (k+6)^2 (k+9)^2} a_0$$

$$y = x^k a_0 \left[ 1 - \frac{x^3}{(k+3)^2} + \frac{x^6}{(k+3)^2 (k+6)^2} - \frac{x^9}{(k+3)^2 (k+6)^2 (k+9)^2} \dots \right] \dots (5)$$

when  $k = 0$

$$y_1 = a_0 \left[ 1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \cdot 6^2} - \frac{x^9}{3^2 \cdot 6^2 \cdot 9^2} \dots \right] \dots (6)$$

Differentiating (5) w.r.t.  $k$

$$\frac{\partial y}{\partial k} = (x^k \log x) a_0 \left[ 1 - \frac{x^3}{(k+3)^2} \right.$$

$$\left. + \frac{x^6}{(k+3)^2 (k+6)^2} \right.$$

$$\left. - \frac{x^9}{(k+3)^2 (k+6)^2 (k+9)^2} \dots \right]$$

$$+ x^k a_0 \left[ \frac{2x^3}{(k+3)^3} - \frac{2x^6}{(k+3)^3 (k+6)^2} - \frac{2x^6}{(k+3)^2 (k+6)^3} \dots \right]$$

putting  $k = 0$   $\left( \frac{\partial y}{\partial k} \right)_{k=0} = y_2$

$$= (\log x) a_0 \left[ 1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \cdot 6^2} - \frac{x^9}{3^2 \cdot 6^2 \cdot 9^2} \dots \right]$$

$$+ a_0 \left[ \frac{2x^3}{3^3} - \frac{2x^6}{3^3 \cdot 6^2} - \frac{2x^6}{3^2 \cdot 6^3} \dots \right] \dots (7)$$

Hence the general solution

$$y = c_1 y_1 + c_2 y_2$$

Q. 4. (b) Establish any two of the following recurrence formulae for the Legendre polynomials

$$(1) P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$(2) (x^2 - 1)P_n'(x) = n(xP_n(x) - P_{n-1}(x))$$

$$(3) (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$$\text{Ans. (1) } P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$\text{Let } v = (x^2 - 1)^n$$

$$\frac{dv}{dx} = n(x^2 - 1)^{n-1} 2x$$

$$(x^2 - 1) \frac{dv}{dx} = n(x^2 - 1)^n 2x = 2nxv$$

$$(1-x^2) \frac{dv}{dx} + 2nxv = 0$$

Differentiating  $(n+1)$  times by Leibnitz theorem,

$$\left[ (1-x^2) \frac{d^{n+2}v}{dx^{n+2}} + (n+1)(-2x) \frac{d^{n+1}v}{dx^{n+1}} + \frac{(n+1)n}{2!} (-2) \frac{d^n v}{dx^n} \right] + 2n \left[ x \frac{d^{n+1}v}{dx^{n+1}} + (n+1) \frac{d^n v}{dx^n} \right] = 0$$

$$(1-x^2) \frac{d^{n+2}v}{dx^{n+2}} - 2x \frac{d^{n+1}v}{dx^{n+1}} + n(n+1) \frac{d^n v}{dx^n} = 0$$

$$(1-x^2) \frac{d^2 v_n}{dx^2} - 2x \frac{dv_n}{dx} + n(n+1)v_n = 0$$

$$\text{Where } v_n = \frac{d^n v}{dx^n}$$

which is Legendre's equation and  $v_n$  is its solution.

$v_n = \frac{d^n}{dx^n} (x^2 - 1)^n$  contains only positive power of  $x$ , it must be a constant multiple of

$$v_n = cP_n(x)$$

$$cP_n(x) = \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{d^n}{dx^n} [(x-1)^n (x+1)^n] \dots (1)$$

$$= (x-1)^n n! + n c_1 n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (x+1)^n + \dots + (x+1)^n n!$$

$$= n!(x+1)^n + \text{terms containing powers of } (x-1)$$

Putting  $x = 1$

$$cP_n(1) = n! 2^n \Rightarrow c = 2^n n!$$

$$(1) \Rightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$(2) (x^2 - 1)P_n'(x) = n(xP_n(x) - P_{n-1}(x))$$

$$\text{Rec. formula } P_n' - xP_{n-1}' = nP_{n-1} \dots (1)$$

$$\text{Rec. formula } xP_n' - P_{n-1}' = nP_n \dots (2)$$

Multiplying (2) by  $x$  and subtracting from (1), we get

$$(1-x^2)P_n' = n[P_{n-1} - xP_n]$$

$$(x^2-1)P_n' = n[xP_n - P_{n-1}]$$

$$(3) (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \dots (1)$$

Differentiating both sides w.r.t  $h$ , we get

$$-\frac{1}{2}(1-2xh+h^2)^{-3/2} (2h-2x)$$

$$= \sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$(x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)$$

$$\sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

$$(x-h) \sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)$$

$$\sum_{n=0}^{\infty} n h^{n-1} P_n(x)$$

Equating the coefficient of  $h^n$  in both sides.

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2nxP_n$$

$$+ (n-1)P_{n-1}$$

$$(n+1)P_n = (2n+1)xP_n - nP_n$$

Q. 4. (c) Prove any two of the following recurrence formulae for the Bessel's functions  $J_n(x)$ :

$$(1) 2n J_n(x) = x(J_{n+1} + J_{n-1})$$

$$(2) J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4) J_{n+4}(x)$$

$$J_{n+4}(x)$$

$$(3) x^2 J_n''(x) = (n^3 - n - x^2) J_n(x) + xJ'(x)_{n+1}$$

Ans. (1)  $2n J_n(x) = x(J_{n+1} + J_{n-1})$

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

$$x^n J_n' + nx^{n-1} J_n = x^n J_{n-1}$$

Dividing by  $x^n$

$$J_n' + \frac{n}{x} J_n = J_{n-1} \quad \dots(1)$$

Again

$$\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}$$

$$x^{-n} J_n' + nx^{-n-1} J_n = -x^{-n} J_{n+1}$$

or  $-J_n' + \frac{n}{x} J_n = J_{n+1} \quad \dots(2)$

Adding (1) & (2), we get,

$$\frac{2n}{x} J_n = J_{n+1} + J_{n-1}$$

$$2n J_n = x(J_{n+1} + J_{n-1})$$

$$(2) J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4) J_{n+4}(x)$$

$$J_{n+4}(x)$$

$$J_n' + \frac{n}{x} J_n = J_{n-1}$$

Put  $n = n + 4$

$$J_{n+4}' + \frac{(n+4)}{x} J_{n+4} = J_{n+3} \quad \dots(1)$$

Again  $-J_n' + \frac{n}{x} J_n = J_{n+1}$

Put  $n = n + 4$

$$-J_{n+4}' + \frac{(n+4)}{x} J_{n+4} = J_{n+5} \quad \dots(2)$$

Adding (1) & (2) we get

$$\frac{2(n+4)}{x} J_{n+4} = J_{n+5} + J_{n+3}$$

$$(3) x^2 J_n''(x) = (n^3 - n - x^2) J_n(x) + xJ'(x)_{n+1}$$

$$xJ_n' = nJ_n - xJ_{n+1} \quad \dots(1)$$

Differentiating

$$xJ_n'' + J_n' = nJ_n' - xJ_{n+1}' - J_{n+1}$$

$$x^2 J_n'' = (n-1)xJ_n' - x^2 J_{n+1}' - xJ_{n+1} \quad \dots(2)$$

Rec. from  $xJ_n' = -nJ_n + xJ_{n-1}$

$$xJ_{n+1}' = -(n+1)J_{n+1} + xJ_n \quad \dots(3)$$

$$(2) \Rightarrow x^2 J_n'' = (n-1)xJ_n'[nJ_n - xJ_{n+1}]$$

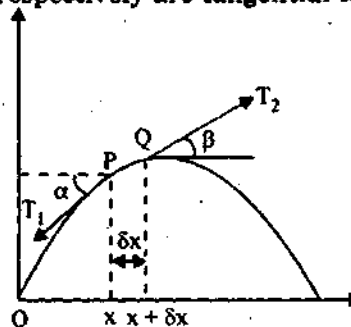
$$-x[-(n+1)J_{n+1} + xJ_n - xJ_{n+1}]$$

$$x^2 J_n'' = (n^2 - n - x^2)J_n + xJ_{n+1}$$

Q. 5. Solve any Two parts of the following: — 10 × 2 = 20

(a) Derive the one dimensional wave equation for vibrating string under suitable conditions.

Ans. One dimensional wave equation: Let  $m$  be the mass per unit length of the string. Consider the motion of an element PQ of length  $\delta s$ , the tension  $T_1$  and  $T_2$  at P and Q respectively are tangential to the curve.



Since there is no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (constant)} \quad \dots(1)$$

Mass of element PQ =  $m\delta s$

By Newton's second law of motion, the equation of motion

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin \beta - T_1 \sin \alpha$$

$$\frac{m\delta s}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s} (\tan \beta - \tan \alpha)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m \delta x} \left[ \left( \frac{\partial y}{\partial x} \right)_{x+\delta x} - \left( \frac{\partial y}{\partial x} \right)_x \right]$$

Since  $\delta s = \delta x$  to a first approximation and  $\tan \alpha$  and  $\tan \beta$  are the slopes of the curve of the string at  $x$  and  $x + \delta x$

$$\text{or } \frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \left[ \frac{\left( \frac{\partial y}{\partial x} \right)_{x+\delta x} - \left( \frac{\partial y}{\partial x} \right)_x}{\delta x} \right]$$

$$\text{or } = \frac{T}{m} \frac{\partial^2 y}{\partial x^2} \text{ as } \delta x \rightarrow 0$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \text{ where } c^2 = \frac{T}{m}$$

The boundary conditions

(i)  $y = 0$  when  $x = 0$  (ii)  $y = 0$  when  $x = l$

If the string is made to vibrate by pulling it into a curve  $y = f(x)$  and then releasing it with the velocity  $g(x)$ , the initial conditions are

(i)  $y = f(x)$  when  $t = 0$

(ii)  $\left( \frac{\partial y}{\partial t} \right) = g(x)$  when  $t = 0$

**Q. 5. (b) Describe the method of separation of variables for solving a partial differential equation. Hence solve the one dimensional heat equation.**

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

under suitable initial and boundary conditions.

**Ans. Method of Separation of variables :** In this method, we take solution which breaks up into a product of functions each of which contains only one of the independent variable.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \text{ or } \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \dots(1)$$

Let  $u = X(x)T(t)$  ... (2) be the solution of equation (1)

Where  $X$  is a function of  $x$  and  $T$  is a function of  $t$  only.

$$XT'' = c^2 X''T$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = k \dots(3)$$

$$\frac{d^2 X}{dx^2} - kX = 0 \text{ and } \frac{dT}{dt} - kc^2 T = 0$$

(i) When  $k$  is positive and  $= p^2$

$$\frac{d^2 X}{dx^2} - p^2 X = 0 \text{ and } \frac{dT}{dt} - p^2 c^2 T = 0$$

$$X = c_1 e^{px} + c_2 e^{-px}, T = c_3 e^{c^2 p^2 t}$$

(ii) When  $k$  is negative and  $= -p^2$

$$\frac{d^2 X}{dx^2} + p^2 X = 0 \text{ and } \frac{dT}{dt} + p^2 c^2 T = 0$$

$$X = c_1 \cos px + c_2 \sin px, T = c_3 e^{-c^2 p^2 t}$$

(iii) when  $k = 0$

$$X = c_1 x + c_2, T = c_3$$

since  $u$  decrease as time  $t$  increases, the solution of heat equation in

$$u = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$$

**Q. 5. (c) Characterize the following partial differential equations into elliptic, parabolic and hyperbolic equations**

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = F(x, y, u, u_x, u_y)$$

Here  $A, B, C$  may be functions of  $x$  and  $y$ .

$$\text{Ans. } A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2}$$

$$= F(x, y, u, u_x, u_y) \dots(1)$$

Equation (1) is

(i) Elliptic if  $(2B)^2 - 4AC < 0$

$$4B^2 - 4AC < 0$$

$$B^2 - AC < 0$$

(ii) Hyperbolic if  $(2B)^2 - 4AC > 0$

$$B^2 - AC > 0$$

(iii) Parabolic if  $(2B)^2 - 4AC = 0$

$$B^2 - AC = 0$$

The differential equations governing the phenomenon of one and two dimensional heat flow are parabolic and elliptic respectively, while the motion of a vibrating string is described by a hyperbolic partial differential equation.