SYLLABUS

ENGINEERING MATHEMATICS – IV

SUBJECT CODE: 10 MAT 41

PART-A

Unit-I: NUMERICAL METHODS - 1

Numerical solution of ordinary differential equations of first order and first degree; Picard's method, Taylor's series method, modified Euler's method, Runge-kutta method of fourth-order. Milne's and Adams - Bashforth predictor and corrector methods (No derivations of formulae).

Unit-II: NUMERICAL METHODS – 2

Numerical solution of simultaneous first order ordinary differential equations: Picard's method, Runge-Kutta method of fourth-order. Numerical solution of second order ordinary differential equations: Picard's method, Runge-Kutta method and Milne's method.

Unit-III: Complex variables – 1

unction of a complex variable, Analytic functions-Cauchy-Riemann equations in cartesian and polar forms. Properties of analytic functions. Application to flow problems- complex potential, velocity potential, equipotential lines, stream functions, stream lines.

Unit-IV: Complex variables – 2

Conformal Transformations: Bilinear Transformations. Discussion of Transformations: $w = z^2$, w = ez , $w = z + (a2 / z)$. Complex line integrals- Cauchy's theorem and Cauchy's integral formula.

PART-B

Unit-V: SPECIAL FUNCTIONS

Solution of Laplace equation in cylindrical and spherical systems leading Bessel's and Legendre's differential equations, Series solution of Bessel's differential equation leading to Bessel function of first kind. Orthogonal property of Bessel functions. Series solution of Legendre's differential equation leading to Legendre polynomials, Rodrigue's formula.

Unit-VI: PROBABILITY THEORY – 1

Probability of an event, empherical and axiomatic definition, probability associated with set theory, addition law, conditional probability, multiplication law, Baye's theorem.

Unit-VII: PROBABILITY THEORY- 2

Random variables (discrete and continuous), probability density function, cumulative density function. Probability distributions – Binomial and Poisson distributions; Exponential and normal distributions.

Unit-VIII: SAMPLING THEORY

Sampling, Sampling distributions, standard error, test of hypothesis for means, confidence limits for means, student's t-distribution. Chi -Square distribution as a test of goodness of fit

Index

Unit I
Numerical Methods I

Numerical Solution of First Order and First Degree Ordinary Differential Equations

The fundamental laws of physics, mechanics, electricity and thermodynamics are usually based on empirical observations that explain variations in physical properties and states of the systems. Rather than describing the state of physical system directly, the laws are usually couched in terms of spatial and temporal changes. The following table gives a few examples of such fundamental laws that are written in terms of the rate of change of variables (*t* = time and *x*= position)

Physical Law	Mathematical Expression	Variables and Parameters
Newton's second law of motion	$\frac{dv}{dt} = \frac{F}{m}$	Velocity(v), force (F) and mass (M)
Fourier's Law of Heat Conduction	$q = -k\frac{dT}{dx}$	Heat flux (q), thermal conductivity(k) and temperature (T)
Faraday's Law (Voltage drop across an inductor)	$\Delta V_L = L\frac{di}{dt}$	Voltage drop (ΔV_L), inductance4 (L) and current (i)

The above laws define mechanism of change. When combined with continuity laws for energy, mass or momentum, differential equation arises. The mathematical expression in the above table is an example of the **Conversion of a Fundamental law to an Ordinary Differential Equation**. Subsequent integration of these differential equations results in mathematical functions that describe the spatial and temporal state of a system in terms of energy, mass or velocity variations. In fact, such mathematical relationships are the basis of the solution for a great number of engineering problems. But, many ordinary differential equations arising in real-world applications and having lot of practical significance cannot be solved exactly using the classical analytical methods. These ode can be analyized qualitatively. However, qualitative analysis may not be able to give accurate answers. A ***numerical method*** can be used to get an accurate approximate solution to a differential equation. There are many programs and packages available for solving these differential equations. With today's computer, an accurate solution can be obtained rapidly. In this chapter we focus on basic numerical methods for solving initial value problems.

Analytical methods, when available, generally enable to find the value of *y* for all values of *x*. Numerical methods, on the other hand, lead to the values of *y* corresponding only to some finite set of values of *x*. That is the solution is obtained as a table of values, rather than as continuous function. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution inevitably involves an error which should be small but may, if it is not controlled, swamp the true solution. Therefore we must be concerned with two aspects of numerical solutions of ODEs: both the method itself and its accuracy.

The general form of first order differential equation, in implicit form, is $F(x, y, y^{/}) = 0$ and in the explicit form is $\frac{dy}{dx} = f(x, y)$. An **Initial Value Problem** (IVP) consists of a differential equation and a condition which the solution much satisfies (or several conditions referring to the same value of x if the differential equation is of higher order). In this chapter we shall consider IVPs of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{1}$$

Assuming f to be such that the problem has a unique solution in some interval containing x_0, we shall discuss the methods for computing numerical values of the solution. These methods are **step-by-step methods**. That is, we start from $y_0 = y(x_0)$ and proceed stepwise. In the first step, we compute an approximate value y_1 of the solution y of (1) at $x = x_1 = x_0 + h$. In the second step we compute an approximate value y_2 of the solution y at $x = x_2 = x_0 + 2h$, etc. Here h is fixed number for example 0.1 or 0.001 or 0.5 depends on the requirement of the problem. In each step the computations are done by the same formula.

The following methods are used to solve the IVP (1).

1. Taylor's Series Method
2. Euler and Modified Euler Method
3. Runge – Kutta Method
4. Milne's Method
5. Adams – Bashforth Method

1. Taylor's Series Method

Consider an IVP $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$. Let us approximate the exact solution y(x) to a power series in $(x - x_0)$ using Taylor's series. The Taylor's series expansion of y(x) about the point $x = x_0$ is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y^{/}(x_0) + \frac{(x - x_0)^2}{2!} y^{//}(x_0) + \frac{(x - x_0)^3}{3!} y^{///}(x_0) + \frac{(x - x_0)^4}{4!} y^{IV}(x_0) + \cdots\cdots \tag{1}$$

From the differential equation, we have $y^{/}(x) = \frac{dy}{dx} = f(x, y)$. Differentiating this successively, we can get $y^{//}(x), y^{///}(x), y^{/V}(x)$, etc. Putting $x = x_0$ and $y = y_0$, the values of $y^{//}(x_0), y^{///}(x_0), y^{/V}(x_0)$, etc. can be obtained. Hence the Taylor's series (1) gives the values of y for every value of x for which (1) converges.

On finding the value of y_1 for $x = x_1$ from (1), $y^{//}(x), y^{///}(x), y^{/V}(x)$, etc. can be evaluated at $x = x_1$ from the differential equation

Problems:

1. Find by Taylor's series method the value of y at x = 0.1 and 0.2 five places of decimals for the IVP $\frac{dy}{dx} = x^2 y - 1, \quad y(0) = 1.$

Soln:

Given $x_0 = 0,\ y_0 = 1$ and $f(x, y) = x^2 y - 1$

Taylor's series expansion about the point $x = 0 (= x_0)$ is

$$y(x) = y(0) + \frac{(x-0)}{1!} y^{/}(0) + \frac{(x-0)^2}{2!} y^{//}(0) + \frac{(x-0)^3}{3!} y^{///}(0) + \frac{(x-0)^4}{4!} y^{/V}(0) + \cdots\cdots$$

i.e. $$y(x) = y(0) + xy^{/}(0) + \frac{x^2}{2} y^{//}(0) + \frac{x^3}{6} y^{///}(0) + \frac{x^4}{24} y^{/V}(0) + \cdots\cdots \qquad (1)$$

It is given that $y(0) = 1$

$\frac{dy}{dx} = y^{/}(x) = x^2 y - 1 \quad \Rightarrow\Rightarrow\Rightarrow \quad y^{/}(0) = -1$

Differentiating $y^{/}(x) = x^2 y - 1$ successively three times and putting x = 0 & y = 1, we get

$y^{//}(x) = 2xy + x^2 y^{/} \quad \Rightarrow\Rightarrow\Rightarrow \quad y^{//}(0) = 0$

$y^{///}(x) = 2y + 4xy^{/} + x^2 y^{//} \quad \Rightarrow\Rightarrow\Rightarrow \quad y^{///}(0) = 2$

$y^{iv}(x) = 6y^{/} + 6xy^{//} + x^2 y^{///} \quad \Rightarrow\Rightarrow\Rightarrow \quad y^{iv}(0) = -6$

Putting the values of $y(0), y^{/}(0), y^{//}(0), y^{///}(0), y^{iv}(0)$ in (1), we get

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6)$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

Hence ***y(0.1) = 0.90033*** and ***y(0.2) = 0.80227.***

2. Employ Taylor's series method to obtain approximate value of y at x = 0.1 and 0.2 for the differential equation $\frac{dy}{dx} = 2y + 3e^x, \quad y(0) = 0$.*Compare the numerical solution obtained with the exact solution.*

Soln:

Given $x_0 = 0,\ y_0 = 0$ and $f(x, y) = 2y + 3e^x$

Taylor's series expansion about the point $x = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \frac{x^4}{24} y^{IV}(0) + \cdots\cdots \qquad (2)$$

It is given that $y(0) = 0$

$$\frac{dy}{dx} = y'(x) = 2y + 3e^x \quad \Rightarrow\Rightarrow\Rightarrow \quad y'(0) = 2y(0) + 3e^0 = 3$$

Differentiating $y'(x) = 2y + 3e^x$ successively three times and putting x = y = 0, we get

$$y''(x) = 2y' + 3e^x \quad \Rightarrow\Rightarrow\Rightarrow \quad y''(0) = 2y'(0) + 3 = 9$$

$$y'''(x) = 2y'' + 3e^x \quad \Rightarrow\Rightarrow\Rightarrow \quad y'''(0) = 2y''(0) + 3 = 21$$

$$y^{iv}(x) = 2y''' + 3e^x \quad \Rightarrow\Rightarrow\Rightarrow \quad y^{iv}(0) = 2y'''(0) + 3 = 45$$

Putting the values of $y(0), y'(0), y''(0), y'''(0), y^{iv}(0)$ in (2), we get

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4$$

Hence,

y(0.1) = 3(0.1)+4.5(0.1)^2+3.5(0.1)^3+1.875(0.1)^4

= 0.3486875

and

y(0.2) = 3(0.2)+4.5(0.2)^2+3.5(0.2)^3+1.875(0.2)^4

= 0.8110.

Exact Soln:

The given differential equation can be written as $\frac{dy}{dx} - 2y = 3e^{x}$ which is Leibnitz's linear differential equation.

Its I.F. is I.F = $e^{-\int 2dx} = e^{-2x}$

Therefore the general solution is,

$$ye^{-2x} = \int 3e^{x}(e^{-2x})dx + c = -3e^{-x} + c$$

$$y = -3e^{x} + ce^{2x} \qquad (3)$$

Using the given initial condition *y = 0 when x = 0* in (3) we get ***c = 3***.

Thus the exact solution is $y = 3\left(e^{2x} - e^{x}\right)$

When *x = 0.1*, the exact solution is ***y(0.1) = 0.348695***

When *x = 0.2*, the exact solution is ***y(0.2) = 0.811266***

The above solutions are tabulated as follows:

x	**Numerical**	**Exact**	**Absolute Error Value**
0.1	*0.3486875*	*0.348695*	*0. 75x10 $^{-5}$*
0.2	*0.8110*	*0.811266*	*0. 266x10 $^{-3}$*

3. Using Taylor's series method solve $\frac{dy}{dx} = x^2 - y, \quad y(0) = 1$ *at 0.1 ≤ x ≤ 0.4. Compare the values with the exact solution.*

Soln:

Given $x_0 = 0,\ y_0 = 1$ and $f(x, y) = x^2 - y$

Taylor's series expansion about the point $x = 0$ is

$$y(x) = y(0) + xy^{/}(0) + \frac{x^2}{2} y^{//}(0) + \frac{x^3}{6} y^{///}(0) + \frac{x^4}{24} y^{/V}(0) + \cdots\cdots \qquad (4)$$

It is given that $\qquad y(0) = 0$

$$\frac{dy}{dx} = y^{/}(x) = x^2 - y \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{/}(0) = (0)^2 - 1 = -1$$

Differentiating $y^{/}(x) = x^2 - y$ successively and putting x =0, y = 1, we get

$$y^{//}(x) = 2x - y^{/} \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{//}(0) = 0y^{/}(0) = 0 - (-1) = 1$$

$$y^{///}(x) = 2 - y^{//} \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{///}(0) = 2 - y^{//}(0) = 1$$

$$y^{iv}(x) = -y^{///} \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y^{iv}(0) = -y^{///}(0) = -1$$

Putting the values of $y(0), y^{/}(0), y^{//}(0), y^{///}(0), y^{iv}(0)$ in (4), we get

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

Hence,

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24}$$

= 0.9051625

$$y(0.2) = 1 - (0.2) + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{24}$$

= 0.8212667.

$$y(0.3) = 1 - (0.3) + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{6} - \frac{(0.3)^4}{24}$$

= 0.7491625.

$$y(0.4) = 1 - (0.4) + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6} - \frac{(0.4)^4}{24}$$

= 0.6896.

Exact Soln:

The given differential equation can be written as $\frac{dy}{dx} + y = x^2$ a linear differential equation.

Its I.F. is I.F = $e^{\int dx} = e^x$

Therefore the general solution is,

$$ye^x = \int e^x (x^2) dx + c = (x^2 - 2x + 2)e^x + c$$

$$y = (x^2 - 2x + 2) + ce^{-x} \qquad (5)$$

Using the given condition *y(0) = 1 in* (5) we get **1 = 2 + c** or ***c = -1***.

Hence the exact solution is $y = (x^2 - 2x + 2) - e^{-x}$

The exact solution at ***x = 0.1, 0.2, 0.3 and 0.4*** are

y(0.1) = 0.9051625,
y(0.2) = 0.8212692,
y(0.3) = 0.7491817 and
y(0.4) = 0.6896799

The above solutions are tabulated as follows:

x	Numerical	Exact	Absolute Error Value
0.1	***0.9051625***	***0.9051625,***	*0*
0.2	***0.8212667***	***0.8212692,***	*0. 25x10 $^{-5}$*
0.3	***0.7491625***	***0.7491817***	*0. 192x10 $^{-4}$*
0.4	***0.6896***	***0.6896799***	*0. 799x10 $^{-4}$*

2. Modified Euler's Method

Consider the IVP $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$. The following two methods can be used to determine the solution at a point $x = x_n = x_0 + nh$.

Euler's Method :

$$y_{n+1}^{E} = y_n + hf(x_n, y_n), \qquad n = 0,1,2,3,\cdots\cdots \tag{1}$$

Modified Euler's Method :

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{E})\right], \qquad n = 0,1,2,3,\cdots\cdots \tag{2}$$

Remark:

1. The formulae (1) and (2) are also known as Euler's Predictor – Corrector formula.
2. When Modified Euler's method is applied to find the solution at a give point, we first find the solution at that point by using Euler's method and the same will be used in the calculation of Modified Euler's method. Also Modified Euler's method has to be applied repeatedly until the solution is stationary.

Problems:

1. Solve $\frac{dy}{dx} = -\frac{y^2}{1+x}, \quad y(0)=1$ by Euler's method by choosing h = 0.1 and h = 0.05. Also solve the same problem by modified Euler's method by choosing h = 0.05. Compare the numerical solution with analytical solution.

Soln:

Analytical solution is : -

$$\frac{dy}{y^2} = -\frac{dx}{1+x} \quad \Rightarrow \quad \frac{1}{y} = \log(1+x) + c$$

Using the condition y(0) = 1, we get c = 1.

Hence the analytical solution is $y = \frac{1}{1+\log(1+x)}$ $\Rightarrow$ y(0.2) = 0.84579

Now by Euler's method, we have $y_{n+1} = y_n - 0.1\left(\frac{y_n^2}{1+x_n}\right)$

$$y_1 = y(0.1) = 1 - 0.1\left(\frac{(1)^2}{1+0}\right) = 0.9$$

$$y_2 = y(0.2) = 0.9 - 0.1\left(\frac{(0.9)^2}{1+0.1}\right) = 0.82636$$

Error = 0.84579 – 0.82636 = 0.01943

Now taking h = 0.05, Euler's method is $y_{n+1} = y_n - 0.05\left(\frac{y_n^2}{1+x_n}\right)$

$$y_1 = y(0.05) = 1.0 - 0.05\left(\frac{(1)^2}{1+0}\right) = 0.95$$

$$y_2 = y(0.1) = 0.95 - 0.05\left(\frac{(0.95)^2}{1+0.05}\right) = 0.90702$$

$$y_3 = y(0.15) = 0.90702 - 0.05\left(\frac{(0.90702)^2}{1+0.1}\right) = 0.86963$$

$$y_4 = y(0.2) = 0.86963 - 0.05\left(\frac{(0.86963)^2}{1+0.15}\right) = 0.83675$$

Error = 0.84579 – 0.83675 = 0.00904

Note that when h = 0.1, Error was 0.01943, which is more.

Now we use modified Euler's method to find y(0.2) with h = 0.05

Euler's Formula is $y_{n+1} = y_n - 0.05\left(\frac{y_n^2}{1+x_n}\right)$, n = 0,1,2 and 3

Modified Euler Formula is $y_{n+1} = y_n - 0.025\left(\frac{y_n^2}{1+x_n} + \frac{y_{n+1}^E}{1+x_{n+1}}\right)$, n = 0,1,2 and 3

Stage – I: Finding y_1 = y(0.05)

From Euler's formula (for n = 0),

$$y_1^E = y(0.05) = 1.0 - 0.05\left(\frac{(1)^2}{1+0}\right) = 0.95$$

From Modified Euler's formula, we have

$$y_1^{(1)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95)^2}{1+0.05}\right) = 0.95351$$

$$y_1^{(2)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95351)^2}{1+0.05}\right) = 0.95335$$

$$y_1^{(3)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95335)^2}{1+0.05}\right) = 0.95336$$

$$y_1^{(4)} = y(0.05) = 1.0 - 0.025\left(\frac{(1)^2}{1+0} + \frac{(0.95336)^2}{1+0.05}\right) = 0.95336$$

Hence $y_1 = y(0.05) = 0.95336$

Stage – II: Finding y_2 = y(0.1)

From Euler's formula (for n = 1), we get

$$y_2^E = y(0.1) = 0.95336 - 0.05\left(\frac{(0.95336)^2}{1+0.05}\right) = 0.91008$$

From Modified Euler's formula, we have

$$y_2^{(1)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91008)^2}{1+0.1}\right) = 0.91286$$

$$y_2^{(2)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91286)^2}{1+0.1}\right) = 0.91278$$

$$y_2^{(2)} = y(0.1) = 0.95336 - 0.025\left(\frac{(0.95336)^2}{1+0.05} + \frac{(0.91278)^2}{1+0.1}\right) = 0.91278$$

Hence $y_2 = y(0.1) = 0.91278$

Stage – III: Finding y_3 = y(0.15)

From Euler's formula (for n = 2), we get

$$y_3^E = y(0.15) = 0.91278 - 0.05\left(\frac{(0.91278)^2}{1+0.1}\right) = 0.87491$$

From Modified Euler's formula (for n = 2), we have

$$y_3^{(1)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87491)^2}{1+0.15}\right) = 0.87720$$

$$y_3^{(2)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87720)^2}{1+0.15}\right) = 0.87712$$

$$y_3^{(3)} = y(0.15) = 0.91278 - 0.025\left(\frac{(0.91278)^2}{1+0.1} + \frac{(0.87712)^2}{1+0.15}\right) = 0.87712$$

Hence $y_3 = y(0.15) = 0.87712$

Stage – IV: Finding y_4 = y(0.2)

From Euler's formula (for n = 3), we get

$$y_4^E = y(0.2) = 0.87712 - 0.05\left(\frac{(0.87712)^2}{1+0.15}\right) = 0.84367$$

From Modified Euler's formula(for n = 3), we have

$$y_4^{(1)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84367)^2}{1+0.2}\right) = 0.84557$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84557)^2}{1+0.2}\right) = 0.84550$$

$$y_4^{(2)} = y(0.2) = 0.87712 - 0.025\left(\frac{(0.87712)^2}{1+0.15} + \frac{(0.84550)^2}{1+0.2}\right) = 0.84550$$

$$\boxed{\textbf{Hence } y_4 = y(0.2) = 0.84550}$$

Error= 0.84579-0.84550=0.00029
Recall that the error from Euler's method is 0.00904

2. Solve the following IVP by Euler's modified method at **0.2 ≤ x ≤ 0.8** with **h = 0.2**:

$$\frac{dy}{dx} = \log_{10}(x+y), \quad y(0) = 2.$$

Soln:

Given Data is: $x_0 = 0, y_0 = 2, h = 0.2$ and $f(x, y) = \log_{10}(x+y)$

To Find: $y_1 = y(x_1) = y(0.2), y_2 = y(x_2) = y(0.4), y_3 = y(x_3) = y(0.6)$

$\& \, y_4 = y(x_4) = y(0.8)$

Stage – I: Finding y_1 = y(0.2)

From Euler's formula (for n = 0),

$$y_1^E = y(0.2) = 2.0 + 0.2\log_{10}(0+2) = 2.0602$$

Now from Modified Euler's formula (for n = 0), we have

$$y_1^{(1)} = y(0.2) = 2.0 + 0.1\left[\log_{10}(0+2) + \log_{10}(0.2 + 2.0602)\right] = 2.0655$$

$$y_1^{(2)} = y(0.2) = 2.0 + 0.1\left[\log_{10}(0+2) + \log_{10}(0.2 + 2.0655)\right] = 2.0656$$

$$y_1^{(3)} = y(0.2) = 2.0 + 0.1\left[\log_{10}(0+2) + \log_{10}(0.2 + 2.0656)\right] = 2.0656$$

$$\boxed{\textbf{Hence } y_1 = y(0.2) = 2.0656}$$

Stage – II: Finding y_2 = y(0.4)

From Euler's formula (for n = 1),

$$y_2^E = y(0.4) = 2.0656 + 0.2\log_{10}(0.2 + 2.0656) = 2.1366$$

Now from Modified Euler's formula (for n = 1), we have

$$y_2^{(1)} = y(0.4) = 2.0656 + 0.1\left[\log_{10}(0.2 + 2.0656) + \log_{10}(0.4 + 2.1366)\right] = 2.1415$$

$$y_2^{(2)} = y(0.4) = 2.0656 + 0.1\left[\log_{10}(0.2 + 2.0656) + \log_{10}(0.4 + 2.1415)\right] = 2.1416$$

$$y_2^{(3)} = y(0.4) = 2.0656 + 0.1\left[\log_{10}(0.2 + 2.0656) + \log_{10}(0.4 + 2.1416)\right] = 2.1416$$

$$\boxed{\textbf{Hence } y_2 = y(0.4) = 2.1416}$$

Stage – III: Finding y_3 = y(0.6)

From Euler's formula (for n = 2),

$$y_3^E = y(0.6) = 2.1416 + 0.2\log_{10}(0.4 + 2.1416) = 2.2226$$

Now from Modified Euler's formula (for n = 2), we have

$$y_3^{(1)} = y(0.6) = 2.1416 + 0.1\left[\log_{10}(0.4 + 2.1416) + \log_{10}(0.6 + 2.2226)\right] = 2.2272$$

$$y_3^{(2)} = y(0.6) = 2.1416 + 0.1\left[\log_{10}(0.4 + 2.1416) + \log_{10}(0.6 + 2.2272)\right] = 2.2272$$

Hence $y_3 = y(0.6) = 2.2272$

Stage – IV: Finding y_4 = y(0.8)

From Euler's formula (for n = 3),

$$y_3^E = y(0.8) = 2.2272 + 0.2\log_{10}(0.6 + 2.2272) = 2.3175$$

Now from Modified Euler's formula (for n = 3), we have

$$y_4^{(1)} = y(0.8) = 2.2272 + 0.1\left[\log_{10}(0.6 + 2.2272) + \log_{10}(0.8 + 2.3175)\right] = 2.3217$$

$$y_4^{(2)} = y(0.8) = 2.2272 + 0.1\left[\log_{10}(0.6 + 2.2272) + \log_{10}(0.8 + 2.3217)\right] = 2.3217$$

Hence $y_4 = y(0.8) = 2.3217$

The solutions at *0.2 ≤ x ≤ 0.8* are tabulated as follows:

x_n	y_n
0.2	$y_1 = y(0.2) = 2.0656$
0.4	$y_2 = y(0.4) = 2.1416$
0.6	$y_3 = y(0.6) = 2.2272$
0.8	$y_4 = y(0.8) = 2.3217$

3.Using modified Euler's method solve the IVP $\frac{dy}{dx} = \sin x + \cos y, \quad y(2.5) = 0$ at x=3.5 in two steps, modifying the solution thrice at each stages. Here x is in radians.

Soln:

Given $x_0 = 2.5, y_0 = 0, h = 0.5$ and $f(x, y) = \sin x + \cos y$

To Find: $y_1 = y(x_1) = y(3.0)$ and $y_2 = y(x_2) = y(3.5)$

Stage – I: Finding y_1 = y(3.0)

From Euler's formula (for n = 0),

$$y_1^E = y(3.0) = 0.0 + 0.5\left[\sin(2.5) + \cos(0)\right] = 0.7992$$

Now from Modified Euler's formula (for n = 0), we have

$$y_1^{(1)} = y(3.0) = 0.0 + 0.25\left[\left\{\sin(2.5) + \cos(0)\right\} + \left\{\sin(3.0) + \cos(0.7992)\right\}\right] = 0.6092$$

$$y_1^{(2)} = y(3.0) = 0.0 + 0.25\left[\left\{\sin(2.5) + \cos(0)\right\} + \left\{\sin(3.0) + \cos(0.6092)\right\}\right] = 0.6399$$

$$y_1^{(3)} = y(3.0) = 0.0 + 0.25\left[\left\{\sin(2.5) + \cos(0)\right\} + \left\{\sin(3.0) + \cos(0.6399)\right\}\right] = 0.6354$$

$$\boxed{\textbf{Hence } y_1 = y(3.0) = 0.6354}$$

Stage – II: Finding y_2 = y(3.5)

From Euler's formula (for n = 1),

$$y_2^E = y(3.5) = 0.6354 + 0.5\left[\sin(3.0) + \cos(0.6354)\right] = 1.10837$$

Now from Modified Euler's formula (for n = 1), we have

$$y_2^{(1)} = y(3.5) = 0.6354 + 0.25\left[\{\sin(3.0) + \cos(0.6354)\} + \{\sin(3.5) + \cos(1.10837)\}\right] = 0.89572$$

$$y_2^{(2)} = y(3.5) = 0.6354 + 0.25\left[\{\sin(3.0) + \cos(0.6354)\} + \{\sin(3.5) + \cos(0.89572)\}\right] = 0.94043$$

$$y_2^{(3)} = y(3.5) = 0.6354 + 0.25\left[\{\sin(3.0) + \cos(0.6354)\} + \{\sin(3.5) + \cos(0.94043)\}\right] = 0.93155$$

Hence $y_2 = y(3.5) = 0.93155$

4. Using modified Euler's method obtain the solution of the differential equation $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$ with the initial condition ***y = 1 at x = 0*** for the range ***0 < x ≤ 0.6*** in steps of 0.2.

Soln:

Given $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = x + \left|\sqrt{y}\right|$

To Find: $y_1 = y(x_1) = y(0.2), y_2 = y(x_2) = y(0.4)$ and $y_3 = y(x_3) = y(0.6)$

The Entire Calculations can be put in the following Tabular Form

x	$Y_1 = f(x_n, y_n)$	$y_n + hf(x_n, y_n)$	$Y_2 = f(x_{n+1}, y_{n+1})$	$y_{n+1} = y_n + \frac{h}{2}(Y_1 + Y_2)$
0.2	1	1.2	1.2954 1.3088 1.3094	1.2295 1.2309 **1.2309**
0.4	1.3094	1.4927	1.6218 1.6345 1.6350	1.5240 1.5253 **1.5253**
0.6	1.6350	1.8523	1.9610 1.9729 1.9734	1.8849 1.8861 **1.8861**

The solution is: $y(0.2) = 1.2309, y(0.4) = 1.5253$ and $y(0.6) = 1.8861$

3. Runge – Kutta Method

The Taylor's series method to solve IVPs is restricted by the difficulty in finding the higher order derivatives. However, Runge – Kutta method do not require the calculations of higher order derivatives. Euler's method and modified Euler's method are Runge – Kutta methods of first and second order respectively.

Consider the IVP $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$. Let us find the approximate value of *y* at $x = x_{n+1}$, *n = 0,1,2,3,…..* of this numerically, using Runge – Kutta method, as follows:

First let us calculate the quantities k_1, k_2, k_3 and k_4 using the following formulae.

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

Finally, the required solution ***y*** is given by

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Problems:

1. Apply Runge – Kutta method, to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y, \quad y(0) = 1$.

Soln:

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x,y) = x + y$

R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (1)

Now

$$k_1 = hf(x_0, y_0) = 0.2 \times [0+1] \qquad = 0.2$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right] \qquad = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right] = 0.2440$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2 \times \left[\left(0 + 0.2\right) + \left(1 + 0.2440\right)\right] = 0.2888$$

Using the values of k_1, k_2, k_3 and k_4 in (1), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6}\left(0.2 + 0.24 + 0.244 + 0.2888\right) = 1.2468$$

Hence the required approximate value of y is 1.2468.

2.Using Runge – Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, \quad y(0) = 1$ *at*

x = 0.2 & 0.4.

Soln:

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

Stage – I: Finding $y_1 = y(0.2)$

R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$ -------- (2)

$$k_1 = hf(x_0, y_0) = 0.2 \times f(0,1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f\left(0.1, 1.1\right) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f\left(0.1, 1.0936\right) = 0.1967$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = 0.2 \times f\left(0.2, 1.1967\right) = 0.1891$$

Using the values of k_1, k_2, k_3 and k_4 in (2), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6}\left(0.2 + 2(0.19672) + 2(0.1967) + 0.1891\right)$$

= 1+0.19599

= 1.19599

Hence the required approximate value of y is 1.19599.

Stage – II: Finding $y_2 = y(0.4)$

We have $x_1 = 0.1, y_1 = 1.19599$ and $h = 0.2$

R – K method (for n = 1) is: $y_2 = y(0.4) = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$ -------- (3)

$$k_1 = hf(x_1, y_1) = 0.2 \times f(0.2, 1.19599) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2 \times f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2 \times f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 \times f(0.4, 1.3753) \qquad = 0.1688$$

Using the values of k_1, k_2, k_3 and k_4 in (3), we get

$$y_2 = y(0.4) = 1.19599 + \frac{1}{6}\left[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688\right]$$

= 1.19599 + 0.1792

= 1.37519

Hence the required approximate value of y is 1.37519.

3. Apply Runge – Kutta method to find an approximate value of y when x = 0.2 with h = 0.1 for the IVP $\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1$. *Also find the Analytical solution and compare with the Numerical solution.*

Soln:

Given: $x_0 = 0, y_0 = 1, h = 0.1$ and $f(x, y) = 3x + \frac{y}{2}$

Stage – I: Finding $y_1 = y(0.1)$

R – K method (for n = 0) is: $y_1 = y(0.1) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$ -------- (4)

$$k_1 = hf(x_0, y_0) = 0.1 \times f(0,1) \qquad = 0.05$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times f(0.05, 1.025) \qquad = 0.06625$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times f(0.05, 1.033125) \qquad = 0.0666563$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times f\,(0.1, 1.0666563) \qquad = 0.0833328$$

Using the values of k_1, k_2, k_3 and k_4, we get

$$y_1 = y(0.1) = 1.0 + \frac{1}{6}\,[0.05 + 2(0.06625) + 2(0.0666563) + 0.0833328]$$

= 1.0 + 0.0665242

= 1.0665242

Hence the required approximate value of y is 1.0665242.

Stage – II: Finding $y_2 = y(0.2)$

We have $x_1 = 0.1, y_1 = 1.0665242$ and $h = 0.1$

R – K method (for n = 1) is: $y_2 = y(0.2) = y_1 + \frac{1}{6}\,[k_1 + 2k_2 + 2k_3 + k_4]$ -------- (5)

$$k_1 = hf\,(x_1, y_1) = 0.1 \times f\,(0.1, 1.0665242) \qquad = 0.0833262$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 \times f\,(0.15, 1.04) \qquad = 0.1004094$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 \times f\,(0.15, 1.0485) \qquad = 0.1008364$$

$$k_4 = hf\,(x_1 + h, y_1 + k_3) = 0.1 \times f\,(0.2, 1.097425) \qquad = 0.1183680$$

Using the values of k_1, k_2, k_3 and k_4 in (5), we get

$$y_2 = y(0.2) = 1.0665242 + \frac{1}{6}\,[0.0833262 + 2(0.1004094) + 2(0.1008364) + 0.1006976]$$

= 1.0665242 + 0.1006976

= 1.1672218

Hence the required approximate value of y is 1.1672218.

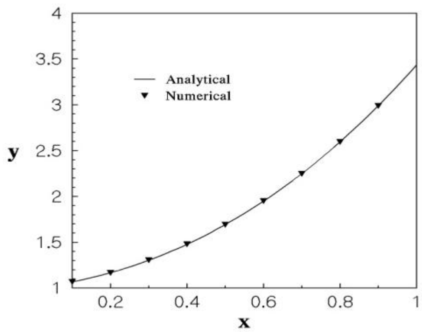
Exact Solution

The given DE can be written as $\frac{dy}{dx} - \frac{y}{2} = 3x$ which is a linear equation whose solution is:

$y = -6x - 12 + 13e^{\frac{x}{2}}$.

The Exact solution at x = 0.1 is y(0.1) = 1.0665242 and at x = 0.2 is y(0.2) = 1.1672218

Both the solutions and the error between them are tabulated as follows:



x	y (Exact)	y (Numerical)
0.1	1.0665243	1.0665242
0.2	1.1672219	1.1672218
0.3	1.3038452	1.3038450
0.4	1.4782359	1.4782357
0.5	1.6923304	1.6923302
0.6	1.9481645	1.9481643
0.7	2.2478782	2.2478779
0.8	2.5937211	2.5937207
0.9	2.9880585	2.9880580
1.0	3.4333766	3.4333761

x_n	y_n (Exact)	y_n (Numerical)	Absolute Error
0.1	1.0665243	1.0665242	0.0000001
0.2	1.1672219	1.1672218	0.0000001

Multi-step Methods:

To solve a differential equation over an interval (x_n, x_{n+1}), using previous single-step methods, only the values of y at the beginning of interval is required. However, in the following methods, four prior values are needed for finding the value of y_{n+1} at a given value of x. Also the solution at y_{n+1} is obtained in two stages. This method of refining an initially crude estimate of y_{n+1} by means of a more accurate formula is known as **Predictor–Corrector method**. A ***Predictor Formula*** is used to predict the value of y_{n+1} and then a ***Corrector Formula*** is applied to calculate a still better approximation of y_{n+1}. Now we study two such methods namely (i) Milne's method and (ii) Adams – Bashforth method.

(I) Milne's Method:

Given $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$. To find an approximate value of y at $x = x_0 + nh$ by Milne's method, we proceed as follows: Using the given value of $y(x_0) = y_0$, we compute $y(x_1) = y(x_0 + h) = y_1, y(x_2) = y(x_0 + 2h) = y_2$ and $y(x_3) = y(x_0 + 3h) = y_3$ using Taylor's series method.

Next, we calculate $f_1 = f(x_1, y_1), f_2 = f(x_2, y_2)$ and $f_3 = f(x_3, y_3)$. Then, the value of y at $x = x_4 = x_0 + 4h$ can be found in the following two stages.

I Stage: Predictor Method

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

Then we compute $f_4 = f(x_4, y_4^{(P)})$

II Stage: Corrector Method

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

Then, an ***improved value of f_4 is computed*** and again, corrector formula is applied to find a better value of y_4. We repeat the step until y_4 remains unchanged.

(II) Adams – Bashforth Method:

Given $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$. Using the given value of $y(x_0) = y_0$, we first compute $y(x_{-1}) = y(x_0 - h) = y_{-1}, y(x_{-2}) = y(x_0 - 2h) = y_{-2}$ and $y(x_{-3}) = y(x_0 - 3h) = y_{-3}$ using Taylor's series method.

Next, we calculate $f_{-1} = f(x_{-1}, y_{-1}), f_{-2} = f(x_{-2}, y_{-2})$ and $f_{-3} = f(x_{-3}, y_{-3})$. Now, the value of y at $x = x_1$ *(or y_1)* can be determined in two stages:

I Stage: Predictor Method

$$y_1^{(P)} = y_0 + \frac{h}{24}\left[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right]$$

Next, we compute $f_1 = f(x_1, y_1^{(P)})$. To find a better approximation to y_1, the following corrector formula is used.

II Stage: Corrector Method

$$y_1^{(C)} = y_0 + \frac{h}{24}\left[f_1 + 19f_0 - 5f_{-1} + f_{-2}\right]$$

Then, an ***improved value of*** $f_1 = f(x_1, y_1^{(C)})$ ***is calculated*** and again, corrector formula is applied to find a better value of y_1. This step is repeated until y_1 remains unchanged and then proceeds to calculate y_2 as above.

Problems:

1. Use Milne's method to find y(0.3) for the IVP $\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1$

Soln:

First, let us find the values of *y* at the points *x = -0.1, x = 0.1* and *x = 0.2* by using Taylor's series method for the given IVP.

Taylor's expansion of *y(x)* about the point $x = 0 (= x_0)$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \qquad \text{--------} \ (1)$$

Given

$$y'(x) = x^2 + y^2 \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y'(0) = 0 + 1 = 1$$

$$y''(x) = 2x + 2yy' \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y''(0) = 2\times1\times1 = 2$$

$$y'''(x) = 2 + 2yy'' + 2(y')^2 \qquad \Rightarrow\Rightarrow\Rightarrow \qquad y'''(0) = 2 + 4 + 2 = 8$$

Using the values of y(0), $y'(0)$, $y''(0)$ and $y'''(0)$ in (1), we get

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3}$$

Putting *x = -0.1, x = 0.1* and *x = 0.2* in the above expression, we get

y(-0.1) = 0.9087, y(0.1) = 1.1113 and y(0.2) = 1.2507

Given:

$x_0 = -0.1, y_0 = 0.9087$ and $f_0 = 0.8357$

$x_1 = 0, y_1 = 1$ and $f_1 = 1$

$x_2 = 0.1, y_2 = 1.1113$ and $f_2 = 1.2449$

$x_3 = 0.2, y_3 = 1.2507$ and $f_3 = 1.6043$

To Find: $y_4 = y(x_4) = y(0.3)$

www.rejinpaul.com

I Stage: Predictor Method

$$y_4^{(P)} = y(0.3) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0.9087 + \frac{4(0.1)}{3}\left[2\times 1) - 1.2449 + 2\times 1.6043\right]$$

= 1.4372

Now we compute $f_4 = f(0.3, 1.4372) = 2.1555$

II Stage: Corrector Method

$$y_4^{(C)} = y(0.3) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$y_4^{(C)} = y(0.3) = 1.1113 + \frac{0.1}{3}\left[1.2449 + (4\times 1.6043) + 2.1555\right]$$

= 1.4386

Now, we compute $f_4 = f(0.3, 1.4386) = 2.1596$

$$y_4^{(C,1)} = y(0.3) = 1.1113 + \frac{0.1}{3}\left[1.2449 + (4\times 1.6043) + 2.1596\right]$$

y(0.3) = 1.43869

Hence, the approximate solution is **y(0.3) = 1.43869**

2. Given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, y(0.2) = 0.02, y(0.4) = 0.0795 and y(0.6) = 0.1762. Compute y(1) using Milne's Method.

Soln:

Stage - I: Finding y(0.8)

Given:

$x_0 = 0, y_0 = 0$ and $f_0 = f(x_0, y_0) = 0$

$x_1 = 0.2, y_1 = 0.02$ and $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, y_2 = 0.0795$ and $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, y_3 = 0.1762$ and $f_3 = f(x_3, y_3) = 0.56895$

To Find: $y_4 = y(x_4) = y(0.8)$

I Stage: Predictor Method

$$y_4^{(P)} = y(0.8) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[2\times 0.1996) - 0.3937_2 + 2\times 0.56895\right]$$

= 0.30491

Now we compute $f_4 = f(0.8, 0.30491) = 0.7070$

II Stage: Corrector Method

$$y_4^{(C)} = y(0.8) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$y_4^{(C)} = y(0.8) = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4\times 0.56895 + 0.7070\right]$$

= 0.3046

Now $f_4 = f(0.8, 0.3046) = 0.7072$

Again applying corrector formula with new f_4, we get

$$y_4^{(C,1)} = y(0.8) = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4\times 0.56895 + 0.7072\right]$$

$\therefore$ **y(0.8) = 0.3046**

Stage - II: Finding y(1.0)

Given:

$x_1 = 0.2, y_1 = 0.02$ and $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, y_2 = 0.0795$ and $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, y_3 = 0.1762$ and $f_3 = f(x_3, y_3) = 0.56895$

$x_4 = 0.8, y_4 = 0.3046$ and $f_4 = f(x_4, y_4) = 0.7072$

To Find: $y_5 = y(x_5) = y(1.0)$

I Stage: Predictor Method

$$y_5^{(P)} = y(1.0) = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 0.02 + \frac{4(0.2)}{3}\left[2\times 0.3937) - 0.56895 + 2\times 0.7072\right]$$

= 0.45544

Now we compute $f_5 = f(1.0, 0.45544) = 0.7926$

II Stage: Corrector Method

$$y_5^{(C)} = y(1.0) = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + f_5\right]$$

$$y_5^{(C)} = y(1.0) = 0.56895 + \frac{0.2}{3}\left[0.56895 + 4\times 0.7072 + 0.7926\right]$$

= 0.4556

Now $f_5 = f(1.0, 0.4556) = 0.7024$

Again applying corrector formula with new f_5, we get

$$y_5^{(C,1)} = y(1.0) = 0.56895 + \frac{0.2}{3}\left[0.56895 + 4\times 0.7072 + 0.7924\right]$$

$\therefore$ **y(1.0) = 0.4556**

3. Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1,\ y(1.1) = 1.233,\ y(1.2) = 1.548,\ y(1.3) = 1.979$ *.Evaluate y(1.4) by Adam's – Bashforth method.*

Soln:

Given:

$$x_{-3} = 1, y_{-3} = 1 \text{ and } f_{-3} = 2$$
$$x_{-2} = 1.1, y_{-2} = 1.233 \text{ and } f_{-2} = 2.70193$$
$$x_{-1} = 1.2, y_{-1} = 1.548 \text{ and } f_{-1} = 3.66912$$
$$x_0 = 1.3, y_0 = 1.979 \text{ and } f_0 = 5.03451$$

To Find: $y_1 = y(x_1) = y(1.4)$

I Stage: Predictor Method

$$y_1^{(P)} = y(1.4) = y_0 + \frac{h}{24}\left[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right]$$

$$= 1.979 + \frac{(0.1)}{24}\left[(55\times 5.03451) - (59\times 3.66912) + (37\times 2.70193) - (9\times 2)\right]$$

= 2.57229

Now we compute $f_1 = f(1.4, 2.57229) = 7.0017$

II Stage: Corrector Method

$$y_1^{(C)} = y(1.4) = y_0 + \frac{h}{24}\left[9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right]$$

$$=1.979+\frac{0.1}{24}\left[9\times 7.0017)+(19\times 5.03451)-(5\times 3.66912)+2.70193\right]$$

y(1.4) = 2.57495

Now, let us compute $f_1 = f(1.4, 2.57495) = 7.0069$

$$y_1^{(C,1)} = 1.979+\frac{0.1}{24}\left[9\times 7.0069)+(19\times 5.03451)-(5\times 3.66912)+2.70193\right]$$

= 2.57514

Again $f_1 = f(1.4, 2.57514) = 7.0073$

$$y_1^{(C,2)} = 1.979+\frac{0.1}{24}\left[9\times 7.0073)+(19\times 5.03451)-(5\times 3.66912)+2.70193\right]$$

y(1.4) = 2.57514

4. Given $\frac{dy}{dx} = x^2 - y, \quad y(0) = 1$. Find *y(0.4)* by Adam's method.

Soln:

First, let us find the values of *y* at the points *x = 0.1, x = 0.2* and *x = 0.3* by using Taylor's series method for the given IVP.

Taylor's expansion of *y(x)* about the point *x = 0(=* x_0*)* is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \qquad \text{-------- (2)}$$

Given

$$y'(x) = x^2 - y \quad \Rightarrow\Rightarrow\Rightarrow \quad y'(0) = 0 - 1 = -1$$

$$y''(x) = 2x - y' \quad \Rightarrow\Rightarrow\Rightarrow \quad y''(0) = 0 - (-1) = 1$$

$$y'''(x) = 2 - 2y'' \quad \Rightarrow\Rightarrow\Rightarrow \quad y'''(0) = 2 - 1 = 1$$

Using the values of y(0), $y'(0)$, $y''(0)$ and $y'''(0)$ in (2), we get

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{3}$$

Putting *x = 0.1, x = 0.2* and *x = 0.3* in the above expression, we get

y(0.1) = 0.9051, y(0.2) = 0.8212 and y(0.3) = 0.7492

Let $x_{-3} = 0, y_{-3} = 1$ and $f_{-3} = -1$

$x_{-2} = 0.1, y_{-2} = 0.9051$ and $f_{-2} = 0.8951$

www.rejinpaul.com

$$x_{-1} = 0.2, y_{-1} = 0.8212 \text{ and } f_{-1} = -0.7812$$
$$x_0 = 0.3, y_0 = 0.7492 \text{ and } f_0 = -0.6592$$

To Find: $y_1 = y(x_1) = y(0.4)$

I Stage: Predictor Method

$$y_1^{(P)} = y(0.4) = y_0 + \frac{h}{24}\left[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right]$$
$$= 0.7492 + \frac{(0.1)}{24}\left[55 \times (-0.6592) - 59 \times (-0.7812) + 37 \times (-0.8951) - 9 \times (-1)\right]$$

= 0.6896507

Now we compute $f_1 = f(0.4, 0.6896507) = -0.5296507$

II Stage: Corrector Method

$$y_1^{(C)} = y(0.4) = y_0 + \frac{h}{24}\left[9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}\right]$$
$$= 0.7492 + \frac{0.1}{24}\left[9 \times (-0.5297) + 19 \times (-0.6592) - 5 \times (-0.7812) - 0.895125\right]$$

y(0.4) = 0.6896522

Unit II
Numerical Methods II

Introduction:

In this unit we discuss numerical solution of simultaneous first order ODEs And also second order ODEs as an extension of some of the earlier discussed method for solving ODEs of first order.

Numerical solution of simultaneous first order ODEs

Picard's method:

Consider the following system of equations:

$$\frac{dy}{dx} = f(x, y, z) \ldots\ldots\ldots\ldots\ldots 1$$

$$\frac{dz}{dx} = g(x, y, z) \ldots\ldots\ldots\ldots\ldots 2$$

With the initial condition $y(x_0)=y_0$

that is, $y = y_0 \text{ and } z = z_0 \text{ at } x = x_0$

we have to find successive approximations for y and z interms of x.

from 1 and 2

$$dy = f(x, y, z)dx$$

where $y=y_0$ at $x=x_0$

$$dz=g(x,y,z)dx$$

where, $z=z_0$ at $x=x_0$

Integrating LHS between y_0 and y RHS between x_0 and x we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y, z)dx$$

Integrating LHS between z_0 and z RHS between x_0 and x we have

$$\int_{z_0}^{z} dz = \int_{x_0}^{x} g(x, y, z)dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y, z)dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y, z)dx$$

$$z - z_0 = \int_{x_0}^{x} g(x, y, z)dx$$

$$z = z_0 + \int_{x_0}^{x} g(x, y, z)dx$$

Problems

1. Use picard's method to find y(0.1) and z(0.1) given that

   $\frac{dy}{dx}$=x+z, $\frac{dz}{dx}$=x-y^{2}and y(0)=2, z(0)=1

   Soln. we have a system of two equation and we need to find two approximations for y and z as functions of x.

$$\frac{dy}{dx} = x + z; y(0) = 2$$

$$\frac{dz}{dx} = x - y^2; z(0) = 1$$

$$\int_2^y dy = \int_0^x (x + z)dx;$$

$$\int_1^z dz = \int_0^x (x - y^2)dx$$

$$y = 2 + \int_0^x (x + z)dx ----(1)$$

$$z = 1 + \int_0^x (x - y^2)dx ----(2)$$

The first approximation for y and z are obtained by replacing the initial values of y and z in the RHS of (1) and (2).

$$y_1 = 2 + \int_0^x (x + 1)dx$$

$$y_1 = 2 + x + \frac{x^2}{2}$$

$$y_2 = 2 + \int_0^x (x + z_1)dx$$

$$= 2 + \int_0^x (x + (1 - 4x + \frac{x^2}{2}))dx$$

$$= 2 + x - \frac{3x^2}{2} + \frac{x^3}{6}$$

$$y(0.1) = 2 + (0.1) - \frac{3(0.1)^2}{2} + \frac{0.1^3}{6}$$

$$= 2.0852$$

$$z_1 = 1 + \int_0^x (x + z^2)dx$$

$$z_1 = 1 - 4x + \frac{x^2}{2}$$

$$z_2 = 1 + \int_0^x (x - y_1^2)dx$$

$$= 1 + \int_0^x (x - (2 + x + \frac{x^2}{2}))dx$$

$$= 1 - 4x - \frac{3x^2}{2} - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$$

$$z(0.1) = 0.584$$

2. Find the second approximation for the solution of the following system equations by applying picard's method $\frac{dx}{dt}$=(x+y)t, $\frac{dy}{dt}$=(x-t)y; x=0, y=1 at t=0.

Soln:we have by data

dx=(x+y)tdt; x=0,t=0; dy=(x-t)y; y=1,t=0

$$\int_0^x dx = \int_0^t (x+y)tdt \quad \text{and} \quad \int_1^y dy = \int_0^t (x-t)ydt$$

$$x = \int_0^t (x+y)tdt$$

$$y = 1 + \int_0^t (x-t)ydt$$

Putting x=0, y=1 in the rhs of 1 and 2 we have

$$x_1 = \int_0^t tdt \qquad y_1 = 1 + \int_0^t -tdt$$

$$x_1 = \frac{t^2}{2} \qquad y_1 = 1 - \frac{t^2}{2}$$

$$x_2 = \int_0^t (x_1 + y_1)tdt \qquad y_2 = 1 + \int_0^t (x_1 - t)y_1 dt$$

$$x_2 = \frac{t^2}{2} \qquad y_2 = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8} - \frac{t^5}{20}$$

are the required second approximation

www.rejinpaul.com

Runge – Kutta Method of fourth order

Consider the system of equations

$$\frac{dy}{dx} = f(x, y, z),$$

$$\frac{dz}{dx} = g(x, y, z);$$

$$y(x_0) = y_0;\, z(x_0) = z_0$$

We compute $y(x_0+h)$ and $z(x_0+h)$

First let us calculate the quantities k_1, k_2, k_3 and k_4 using the following formulae.

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \qquad l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \qquad l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3, z_0 + l_3\right) \qquad l_4 = hg\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$$

Finally, the required solution ***y*** is given by

$$y(x_0 + h) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

And

$$z(x_0 + h) = z_0 + \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

1. Apply Runge – Kutta method to find an approximate value of y when x = 0.2 with h = 0.1 for the IVP $\frac{dy}{dx} = 3x + \frac{y}{2},\quad y(0) = 1$*. Also find the Analytical solution and compare with the Numerical solution.*

Soln:

Given: $x_0 = 0, y_0 = 1, h = 0.1$ and $f(x, y) = 3x + \frac{y}{2}$

Stage – I: Finding $y_1 = y(0.1)$

R – K method (for n = 0) is: $y_1 = y(0.1) = y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$ -------- (4)

$k_1 = hf(x_0, y_0) = 0.1 \times f(0,1)$ = 0.05

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\times f(0.05, 1.025) \qquad = 0.06625$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\times f(0.05, 1.033125) \qquad = 0.0666563$$

$$k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\times f(0.1, 1.0666563) \qquad = 0.0833328$$

Using the values of k_1, k_2, k_3 and k_4, we get

$$y_1 = y(0.1) = 1.0 + \frac{1}{6}(0.05 + 2(0.06625) + 2(0.0666563) + 0.0833328)$$

$= 1.0 + 0.0665242$

$= 1.0665242$

Hence the required approximate value of y is 1.0665242.

Stage – II: Finding $y_2 = y(0.2)$

We have $x_1 = 0.1, y_1 = 1.0665242$ and $h = 0.1$

R – K method (for n = 1) is: $y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (5)

$$k_1 = hf(x_1, y_1) = 0.1\times f(0.1, 1.0665242) \qquad = 0.0833262$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1\times f(0.15, 1.04) \qquad = 0.1004094$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1\times f(0.15, 1.0485) \qquad = 0.1008364$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1\times f(0.2, 1.097425) \qquad = 0.1183680$$

Using the values of K_1,K_2,K_3 and K_4in (5), we get

$$y_2 = y(0.2) = 1.0665242 + \frac{1}{6}\ 0.0833262 + 2(0.1004094) + 2(0.1008364) + 0.1006976$$

$= 1.0665242 + 0.1006976$

$= 1.1672218$

Hence the required approximate value of y is 1.1672218.

2 Apply Runge – Kutta method, to find an approximate value of y when x = 0.2 given that. $\frac{dy}{dx} = x + y, \quad y(0) = 1$

Soln: Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = x + y$

R – K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ -------- (1)

Now

$k_1 = hf(x_0, y_0) = 0.2 \times [0 + 1]$ = 0.2

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right]$ = 0.2400

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right]$ = 0.2440

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times [(0 + 0.2) + (1 + 0.2440)]$ = 0.2888

Using the values of k_1, k_2, k_3 and k_4 in (1), we get

$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 0.24 + 0.244 + 0.2888)$ = 1.2468

Hence the required approximate value of y is 1.2468.

Numerical solution of secand order ODEs by Picard's method and Runge kutta method

We present the method explicitly

Let $y'' = g(x, y, y')$ with intial condition $y(x_0) = y_0$, $y'(x_0) = y_0'$ be the second order differential equations.

Now let $y' = z$ this gives $y'' = \frac{dz}{dx}$

The second order differential equations assumes the form $\frac{dz}{dx} = g(x, y, z)$ with the condition $y(x_0) = y_0$ and $z(x_0) = z_0$ where y'_0 is be denoted by z_0

Hence we have two first order simultaneous ODEs

$\frac{dy}{dx} = z$ and $\frac{dz}{dx} = g(x, y, z)$ with $y(x_0) = y_0$; $z(x_0) = z_0$

Taking f(x,y,z)=z we now have the following system of equations for solving

$\frac{dy}{dx} = f(x, y, z)$, $\frac{dz}{dx} = g(x, y, z)$, $y(x_0) = y_0$ and $z(x_0) = z_0$

Milne's Method:

Given $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$. To find an approximate value of *y* at *x=x₀+nh* by Milne's method, we proceed as follows: Using the given value of $y(x_0) = y_0$, we compute $y(x_1) = y(x_0 + h) = y_1, y(x_2) = y(x_0 + 2h) = y_2$ and $y(x_3) = y(x_0 + 3h) = y_3$ using Taylor's series method.

Next, we calculate $f_1 = f(x_1, y_1), f_2 = f(x_2, y_2)$ and $f_3 = f(x_3, y_3)$. Then, the value of y at $x = x_4 = x_0 + 4h$ can be found in the following two stages.

I Stage: Predictor Method

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

Then we compute $f_4 = f(x_4, y_4^{(P)})$

II Stage: Corrector Method

$$y_4^{(C)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

Then, an ***improved value of f_4 is computed*** and again, corrector formula is applied to find a better value of y_4. We repeat the step until y_4 remains unchanged.

1 Use Milne's method to find y(0.3) for the IVP $\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1$

Soln:

First, let us find the values of *y* at the points *x = -0.1, x = 0.1* and *x = 0.2* by using Taylor's series method for the given IVP.

Taylor's expansion of *y(x)* about the point $x = 0 (= x_0)$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \qquad \text{-------- (1)}$$

Given

$$y'(x) = x^2 + y^2 \quad \Rightarrow\Rightarrow\Rightarrow \quad y'(0) = 0 + 1 = 1$$
$$y''(x) = 2x + 2yy' \quad \Rightarrow\Rightarrow\Rightarrow \quad y''(0) = 2 \times 1 \times 1 = 2$$
$$y'''(x) = 2 + 2yy'' + 2(y')^2 \quad \Rightarrow\Rightarrow\Rightarrow \quad y'''(0) = 2 + 4 + 2 = 8$$

Using the values of y(0), $y'(0)$, $y''(0)$ and $y'''(0)$ in (1), we get

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3}$$

Putting *x = -0.1, x = 0.1* and *x = 0.2* in the above expression, we get

y(-0.1) = 0.9087, y(0.1) = 1.1113 and y(0.2) = 1.2507

Given:

$x_0 = -0.1, y_0 = 0.9087$ and $f_0 = 0.8357$

$x_1 = 0, y_1 = 1$ and $f_1 = 1$

$x_2 = 0.1, y_2 = 1.1113$ and $f_2 = 1.2449$

$x_3 = 0.2, y_3 = 1.2507$ and $f_3 = 1.6043$

To Find: $y_4 = y(x_4) = y(0.3)$

I Stage: Predictor Method

$$y_4^{(P)} = y(0.3) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0.9087 + \frac{4(0.1)}{3}\left[(2\times 1) - 1.2449 + 2\times 1.6043\right]$$

= 1.4372

Now we compute $f_4 = f(0.3, 1.4372) = 2.1555$

II Stage: Corrector Method

$$y_4^{(C)} = y(0.3) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$y_4^{(C)} = y(0.3) = 1.1113 + \frac{0.1}{3}\left[1.2449 + (4\times 1.6043) + 2.1555\right]$$

= 1.4386

Now, we compute $f_4 = f(0.3, 1.4386) = 2.1596$

$$y_4^{(C,1)} = y(0.3) = 1.1113 + \frac{0.1}{3}\left[1.2449 + (4\times 1.6043) + 2.1596\right]$$

y(0.3) = 1.43869

Hence, the approximate solution is **y(0.3) = 1.43869**

2 Given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, *y(0.2) = 0.02, y(0.4) = 0.0795 and y(0.6) = 0.1762. Compute y(1) using Milne's Method.*

Soln:

Stage - I: Finding y(0.8)

Given:

$x_0 = 0, y_0 = 0$ and $f_0 = f(x_0, y_0) = 0$

$x_1 = 0.2, y_1 = 0.02$ and $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, y_2 = 0.0795$ and $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, y_3 = 0.1762$ and $f_3 = f(x_3, y_3) = 0.56895$

To Find: $y_4 = y(x_4) = y(0.8)$

I Stage: Predictor Method

$$y_4^{(P)} = y(0.8) = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 0 + \frac{4(0.2)}{3}\left[(2\times 0.1996) - 0.3937_2 + 2\times 0.56895\right]$$

= 0.30491

Now we compute $f_4 = f(0.8, 0.30491) = 0.7070$

II Stage: Corrector Method

$$y_4^{(C)} = y(0.8) = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$y_4^{(C)} = y(0.8) = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4\times 0.56895 + 0.7070\right]$$

= 0.3046

Now $f_4 = f(0.8, 0.3046) = 0.7072$

Again applying corrector formula with new f_4, we get

$$y_4^{(C,1)} = y(0.8) = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4\times 0.56895 + 0.7072\right]$$

$\therefore$ **y(0.8) = 0.3046**

Stage - II: Finding y(1.0)

Given:

$x_1 = 0.2, y_1 = 0.02$ and $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4, y_2 = 0.0795$ and $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6, y_3 = 0.1762$ and $f_3 = f(x_3, y_3) = 0.56895$

$x_4 = 0.8, y_4 = 0.3046$ and $f_4 = f(x_4, y_4) = 0.7072$

To Find: $y_5 = y(x_5) = y(1.0)$

I Stage: Predictor Method

$$y_5^{(P)} = y(1.0) = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 0.02 + \frac{4(0.2)}{3}\left[(2\times 0.3937) - 0.56895 + 2\times 0.7072\right]$$

= 0.45544

Now we compute $f_5 = f(1.0, 0.45544) = 0.7926$

II Stage: Corrector Method

$$y_5^{(C)} = y(1.0) = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + f_5\right]$$

$$y_5^{(C)} = y(1.0) = 0.56895 + \frac{0.2}{3}\left[0.56895 + 4\times 0.7072 + 0.7926\right]$$

= 0.4556

Now $f_5 = f(1.0, 0.4556) = 0.7024$

Again applying corrector formula with new f_5, we get

$$y_5^{(C,1)} = y(1.0) = 0.56895 + \frac{0.2}{3}\left[0.56895 + 4\times 0.7072 + 0.7924\right]$$

$\therefore$ **y(1.0) = 0.4556**

UNIT-III
COMPLEX ANALYSIS-1

INTRODUCTION:

An extension of the concept of real numbers to accommodate complex numbers was evolved while considering solutions of equations like $x^2+1=0$. This equation cannot be satisfied for any real value of x. In fact, the solution of the equation $x^2+1=0$ is of the form $x=\pm\sqrt{-1}$. The square root of -1 cannot be a real no. because the square o any real no. is nonnegative. Similarly, there are any number of algebraic equations whose solutions involve square roots of negative numbers.

FUNCTION OF A COMPLEX VARIABLE:

If z = x+iy is a complex variable, then w = f(z) is called function of a complex variable. W = f(z) = u+iv where u = u(x,y), v = v(x,y). Hence for every point of (x,y) in z-plane, there corresponds (u,v) in w-plane

LIMIT OF A COMPLEX FUNCTION:

Complex value function f(z) defined in the neighbourhood of a point z_0 is said to have limit L as $z\to z_0$ if for all ε>0 however small, there exists a positive real number δ such that

$$|f(z)-L|<\varepsilon \text{ whenever } |z-z_0|<\delta,$$

$$\text{i.e., } \lim_{z\to z_0} f(z)=L$$

CONTINUITY:

A function f(z) is said to be continuous at a point z_0 if $\lim_{z\to z_0} f(z)=f(z_0)$

DIFFERENTIABILITY:

A function f(z) is said to be differentiable at a point z_0 if $\lim_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$ exists and this is unique

ANALYTIC FUNCTION:

A function f(z) is said to be analytic at a point z_0 if f(z) is differentiable at z_0 as well as at all points in a neighbourhood of z_0.

i.e., $f'(z)=\lim_{\delta z\to 0}\dfrac{f(z+\delta z)-f(z)}{\delta z}$ exists and unique for all points in complex region.

NOTE:

Analytic function is also called as regular function or holomorphic function.

CAUCHY'S RIEMANN EQUATIONS OR C-R EQUATIONS IN CARTESIAN FORM:

Statement: If w = f(z) = u+iv is analytic function at the point z = x+iy, then there exists partial derivatives and satisfy the equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ called C-R equations in cartesian form.

Proof: By data f(z) is analytic at a point z = x+iy, there by definition of analytic function,

$$f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}\ \ldots\ldots\ldots\ldots\ldots(1)\text{exists and unique.}$$

We have f(z) = u(x,y) +iv(x,y)

$$f(z+\delta z)=u(x+\delta x,y+\delta y)+iv(x+\delta x,y+\delta y)$$

$$z=x+iy$$

$$\delta z=\delta x+i\delta y$$

Substituting the above in (1) we get

$$f'(z)=\lim_{\delta z\to 0}\frac{\left[u(x+\delta x,y+\delta y)+iv(x+\delta x,y+\delta y)\right]-\left[u(x,y)+iv(x,y)\right]}{\delta x+i\delta y}\ldots\ldots\ldots\ldots(2)$$

Since $\delta z\to 0$, we have 2 possibilities.

Case(i): If δz is only real, then δy = 0

i.e., if $\delta z\to 0$ then $\delta x\to 0$

equn(2) becomes

$$f'(z)=\lim_{\delta x\to 0}\frac{\left[u(x+\delta x,y)-u(x,y)\right]}{\delta x}+i\frac{\left[v(x+\delta x,y)-v(x,y)\right]}{\delta x}$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\ldots\ldots\ldots\ldots\ldots(3)$$

Case(ii): If δz is only imaginary, then δx = 0

i.e., if $\delta z\to 0$ then $\delta y\to 0$

equn(2) becomes

$$f'(z)=\frac{1}{i}\lim_{\delta y\to 0}\frac{\left[u(x,y+\delta y)-u(x,y)\right]}{\delta y}+\lim_{\delta y\to 0}\frac{\left[v(x,y+\delta y)-v(x,y)\right]}{\delta y}$$

$$f'(z)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\ldots\ldots\ldots\ldots\ldots(4)$$

Comparing real and imaginary parts of equations (3) and (4) we get

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \ \&\ \ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\ \ldots\ldots\ldots\ldots\text{proved}$$

CAUCHY'S RIEMANN EQUATIONS OR C-R EQUATIONS IN POLAR FORM:

Statement: If w = f(z) = u+iv is analytic function at the point $z = re^{i\theta}$, then there exists partial derivatives and satisfy the equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ called C-R equations in polar form.

Proof: By data f(z) is analytic at a point $z = re^{i\theta}$, there by definition of analytic function,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \text{ ....(1)exists and unique.}$$

We have f(z) = u(r,θ) +iv(r,θ)

$$f(z+\delta z) = u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)$$

$$\delta z = \delta r e^{i\theta} + ire^{i\theta}\delta\theta$$

Substituting the above in (1) we get

$$f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{\delta r e^{i\theta} + ire^{i\theta}\delta\theta} \text{...........(2)}$$

Since $\delta z \to 0$, we have 2 possibilities.

Case(i): If δz is only real, then δθ = 0

i.e., if $\delta z \to 0$ then $\delta r \to 0$

equn(2) becomes

$$f'(z) = e^{-i\theta} \lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) - u(r,\theta)]}{\delta r} + ie^{-i\theta} \lim_{\delta r \to 0} \frac{[v(r+\delta r, \theta) - v(r,\theta)]}{\delta r}$$

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{...............(3)}$$

Case(ii): If δz is only imaginary, then δr = 0

i.e., if $\delta z \to 0$ then $\delta\theta \to 0$

equn(2) becomes

$$f'(z) = \frac{e^{-i\theta}}{ir} \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{\delta\theta} + i\frac{e^{-i\theta}}{ir} \lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{\delta\theta}$$

$$f'(z) = \frac{e^{-i\theta}}{ir}\left[\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right] \text{...............(4)}$$

Comparing real and imaginary parts of equations (3) and (4) we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \text{ ....proved}$$

HARMONIC FUNCTION:

A function u is said to be harmonic function if it satisfies the Laplace equation.

i.e., $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=o$ in Cartesian form

$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=o$ in polar form.

Theorem:

Statement: The real and imaginary parts of an analytic function are harmonic.

Proof: Let f(z) = u(x,y)+iv(x,y)

Since f(z) is analytic, satisfies C-R equations

i.e., $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ ……..(1) & $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ …………(2)

differentiating (1) partially w.r.t x and (2) w.r.t y , we get

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$ & $\frac{\partial^2 v}{\partial y\partial x}=-\frac{\partial^2 u}{\partial y^2}$

Therefore $\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow$ u satisfies Laplace equation.

Hence real part u is harmonic.

differentiating (1) partially w.r.t y and (2) w.r.t x , we get

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 v}{\partial x^2}$ & $\frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 u}{\partial x\partial y}$

Equating the above equations $\frac{\partial^2 v}{\partial y^2}=-\frac{\partial^2 v}{\partial x^2}$

$\Rightarrow \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \Rightarrow$ vsatisfies Laplace equation.

Hence imaginary part v is harmonic.

PROBLEMS:

1. **Prove that** $u=e^{2x}(x\cos 2y-y\sin 2y)$ **is harmonic. Find the analytic function f(z)**

Soln:

$$\frac{\partial u}{\partial x} = e^{2x}\cos 2y + 2e^{2x}\left(x\cos 2y - y\sin 2y\right)$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}\cos 2y + 4e^{2x}\left(x\cos 2y - y\sin 2y\right)$$

$$\frac{\partial u}{\partial y} = e^{2x}\left(-2x\sin 2y - \sin 2y - 2y\cos 2y\right)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{2x}\left(-4x\cos 2y - 4\cos 2y + 4y\sin 2y\right)$$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{2x}\left(-4x\cos 2y - 4\cos 2y + 4y\sin 2y\right) +$

$$4e^{2x}\cos 2y + 4e^{2x}\left(x\cos 2y - y\sin 2y\right) = 0$$

Therefore u is harmonic.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} ....by C-R equations$$

$$f'(z) = e^{2x}\cos 2y + 2e^{2x}\left(x\cos 2y - y\sin 2y\right) - ie^{2x}\left(-2x\sin 2y - \sin 2y - 2y\cos 2y\right)$$

Put x = z and y=0

$$f'(z) = e^{2z}\left(1 + 2z\right) - ie^{2z}\left(0\right) = e^{2z}\left(1 + 2z\right)$$

Integrating we get

$$f(z) = (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + c$$

$$f(z) = ze^{2z} + ic$$

2)Find the analytic function f(z) where imaginary part is $e^{x}(x\sin y + y\cos y)$

$$Soln: v = e^{x} \;\; x\sin y + y\cos y$$

$$\frac{\partial v}{\partial x} = e^{x} \;\; x\sin y + y\cos y \;\; + e^{x}\sin y$$

$$\frac{\partial v}{\partial y} = e^{x} \;\; x\cos y + \cos y - y\sin y$$

$$\text{we have } f^{1}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\text{By C-R equation } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$: f^{1}(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f^{1}(z) = e^{x}\left[\;\; x\cos y + \cos y - y\sin y \;\; + i \;\; x\sin y = y\cos y + \sin y \right]$$

$$Put \;\; x = z, \; y = 0$$

$$f^{1}(z) = e^{z} \left[z + 1 \right] + i \left[0 \right]$$

$$= (z+1)e^{z}$$

Integrating w.r.t z

$$f(z) = (z+1)e^{z} - (1)e^{z} + c$$

$$f(z) = ze^{z} + c$$

3)Find the analytic function where real part is $e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]$

Solution:

$$u=e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]$$

$$\frac{\partial u}{\partial x}=e^x\left[\ x^2-y^2\ \ \cos y-2xy\sin y\right]+e^x\ \ 2x\cos y-2y\sin y$$

$$\frac{\partial u}{\partial y}\big|(z.0)=e^z\left[z^2+2z\right]$$

$$\frac{\partial u}{\partial y}=e^x\left[-2y\cos y-(x^2-y^2)\sin y-2x\sin y-2xy\cos y\right]$$

$$\frac{\partial u}{\partial y}\Big|_{z,o}=e^z\ \ 0\ =0$$

$$f^1(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\Big|_{z,o}$$

$$=\frac{\partial u}{\partial x}+i\left(-\frac{\partial u}{\partial y}\right)\Big|_{z,o}\ \left[\text{By C-R equations}\right]$$

$$=\left(z^2+2z\right)e^z+0$$

$$f^1(z)=\ z^2+2z\ \ e^z$$

$$f(z)=\int\ z^2+2z\ \ e^z dz$$

$$=\ z^2+2z\ \ e^z-\ \ 2z+2\ \ e^z+(2)e^z+c$$

$$f(z)=z^2e^z+c$$

4) If f(z)=u(x,y)+iv(x,y) is analytic, Show that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\left|f(z)\right|^2=4\left|f^1(z)\right|^2$

Solution:
Let f(z) =u+iv is analytic:

$f^1(\text{z})\,\text{exist}$

$$f^1(\text{z})=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\left|f^1(\text{z})\right|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2=\left(\frac{\partial v}{\partial x}\right)^2}$$

$$\left|f^1(\text{z})\right|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

$$Also\left|f^1(\text{z})\right|^2=u^2+v^2$$

Diff partially (2) w.r.t x

$$\frac{\partial}{\partial x}\left|f(z)\right|=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

Differentialing again w.r.t x we get

$$\frac{\partial^2}{\partial x^2}\left|f(z)\right|^2=2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly we can obtain

$$\frac{\partial^2}{\partial y^2}\left|f(z)\right|^2=2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$(3)+(4)=)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2=2\left[u\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}+v\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\}+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Since u & v are harmonic & using C-R equation

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$, we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left|f(z)\right|^2=2\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$=4\left|f^1(\text{z})\right|^2$$

UNIT-IV
COMPLEX ANALYSIS-2

CONFORMAL TRANSFORMATION:

Let c_1 and c_2 be any two curves in the z-plane intersecting at z0, suppose the transformation w = f(z) transforms the curves c_1 and c_2 to the curves $c_1' \, and \, c_2'$ respectively which intersect at a point $w_0 = f(z_0)$ in the w-plane. Then the transformation is said to be conformal if the angle between c_1 and c_2 is equal to the angle between $c_1' \, and \, c_2'$ in both magnitude and direction.

SOME STANDARD TRANSFORMATION:

1. **Discuss the transformation** $w = z^2$

$$w = z^2 \text{.............}(1)$$
$$u + iv = (x + iy)^2$$
$$= (x^2 - y^2) + i2xy$$
$$\therefore \; u = x^2 - y^2 \;\; ; \;\; v = 2xy \text{...........}(2)$$

case(i): consider a line parallel to y-axis i.e., x=a, a is a constant

eqn(2) becomes

$$u = a^2 - y^2 \;,\; v = 2ay$$
$$\text{or } v^2 = 4a^2y^2 = 4a^2(a^2 - u) = -4a^2(u - a^2)$$

This represents parabola in the w-plane along negative u-axis vertex at the point $(a^2, 0)$

Case(ii): consider a straight line parallel to x-axis i.e., y = b, b is a constant $u = x^2 - b^2$, v=2bx

Or $v^2 = 4b^2x^2 = 4b^2(u + b^2)$

This represents parabola in the w-plane along positive u axis, vertes at the point $(-b^2, 0)$

CONCLUSION: Straight line parallel to coordinate axes in z – plane transforms parabolas in w-plane under the transformation $w = z^2$

\

2. **Discuss the transformation** $w = e^z$

$$w = e^z$$
$$u + iv = e^{(x+iy)} = e^x(\cos y + i \sin y)$$
$$u = e^x \cos y,$$
$$v = e^x \sin y \text{............}(1)$$

Squaring and adding (1)

$u^2 + v^2 = e^{2x}$............(2)

Dividing

$\frac{v}{u} = \tan y$................(3)

Case(i): consider a straight line parallel to x-axis i.e., x = a, a is a constant

Therefore eqn(2 becomes)

$u^2 + v^2 = e^{2a} = r^2$

This represents a circle with centre origin and radius r in the w-plane

Case(ii): consider a straight line parallel to x-axis i.e., y = b, b is a constant

Therefore eqn(2 becomes)

$\frac{v}{u} = \tan b. = m(say)$

$v = um$

This represents a straing line passing thro' the origin in the w-plane

CONCLUSION: Straight line parallel to coordinate axes in z – plane transforms circle with centre origin and straight line passing thro' origin respectively in w-plane under the transformation $w = e^z$

3. Discuss the transformation $w = z + \frac{k^2}{z}, z \neq 0$. What are the points at which the transformation is not conformal.

$f(z) = z + \frac{k^2}{z}$

$f'(z) = 1 - \frac{k^2}{z^2} \neq 0$

Only if $z \neq \pm k$

Therefore the transformation $w = z + \frac{k^2}{z}, z \neq 0$ is not conformal at $z = \pm k$

$w = z + \frac{k^2}{z}$

$w = re^{i\theta} + \frac{k^2}{re^{i\theta}}$

\

$w = r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta)$

$w = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$

$$u = \left(r + \frac{k^2}{r}\right)\cos\theta........(1)$$

$$v = \left(r - \frac{k^2}{r}\right)\sin\theta........(2)$$

Eliminating θ from (1)and (2)

$$\frac{u}{\left(r + \frac{k^2}{r}\right)} = \cos\theta, \qquad \frac{v}{\left(r - \frac{k^2}{r}\right)} = \sin\theta,$$

Squaring and adding we get

$$\frac{u^2}{\left(r + \frac{k^2}{r}\right)^2} + \frac{v^2}{\left(r - \frac{k^2}{r}\right)^2} = 1...............(3)$$

Eliminating r from (1) and (2)

$$\frac{u}{\cos\theta.} = \left(r + \frac{k^2}{r}\right), \qquad \frac{v}{\sin\theta} = \left(r - \frac{k^2}{r}\right)$$

Squaring and subtracting we get

$$\frac{u^2}{\cos^2\theta.} - \frac{v^2}{\sin^2\theta.} = 4k^2$$

$$\frac{u^2}{(2k\cos\theta)^2.} - \frac{v^2}{(2k\sin\theta)^2..} = 1.................(4)$$

case(i): $Let|z| = c_1$

$i.e., r = c_1$

This represent a circle with centre origin and radius c_1

Eqn(3) becomes

$$\frac{u^2}{\left(c_1 + \frac{k^2}{c_1}\right)^2} + \frac{v^2}{\left(c_1 - \frac{k^2}{c_1}\right)^2} = 1$$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Where a= $c_1 + \frac{k^2}{c_1}$ and b $= c_1 - \frac{k^2}{c_1}$

This represent an ellipse in w- plane with foci $(\pm\sqrt{a^2-b^2},0)$ i.e., $(\pm 2k,0)$

Case(ii): Let ampZ= $\theta = c_2$, constant

This represents a straight line passing thro' origin in z-plane

Eqn(4) becomes

$$\frac{u^2}{(2k\cos c_2)^2.} - \frac{v^2}{(2k\sin c_2)^2} = 1$$

$$i.e., \frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$$

Where A= $2k\cos c_2$ and B = $2k\sin c_2$

This represents a hyperbola in w-plane with foci at $(\pm\sqrt{A^2+B^2},0)$ i.e., $(\pm 2k,0)$

CONCLUSION: The circle centred at origin with radius constant in z-plane transforms to an ellipse with foci $(\pm 2k,0)$ in w-plane and straight line passing thro' origin in z-plane transforms to a hyperbola with foci $(\pm 2k,0)$ in w-plane. Since both of these conics have the same foci $(\pm 2k,0)$, they are called confocal conics.

PROBLEM: Find the image if lines parallel to x-axis & lines parallel to y-axis under the transformation $w = z^2$ **Draw neat sketh**

Solution:

$$w = z^2 = (x+iy)^2 = x^2 - y^2 + i(2xy)$$

$$u = x^2 - y^2 \ \& \ v = 2xy$$

Case: Consider a line parallel to y-axis i.e, x=a, a is Constant

$$=) \ u = a^2 - y^2 \ \& \ v = 2ay$$

$$y^2 = 4a^2y^2$$

$$= 4a^2(a^2 - u)$$

$$v^2 = -4a^2(u - a^2)$$

This represent parabola in the w-plane along-ve u-axis vertex at the pt(a^2 ,0)

Case(ii): Consider a straight line 11^{el} to y-a xis i.e, y=b, b is a constant

(1)=) $u = x^2 - b^2 \ \& \ v = 2xb$

$$v^2 = 4x^2b^2 = 4b^2(u + b^2)$$

This represents parabola in the w-plan along the u-axis, vertex at the pt $-b^2, 0$

Hence, we conclude that straight line 11^{el} to Co-ordinate axes in z-plane transforms parabolas in w-plane under $w = z^2$

BILINEAR TRANSFORMATION OR MOBIUS TRANSFORMATION

The transformation $w = \dfrac{az+b}{cz+d}$, where a,b,c,d are real or complex constants such that $ad - bc \neq 0$ is called a bilinear transformation

CROSS RATIO:

Cross ratio of four points z_1, z_2, z_3, z_4 defined by $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$

INVARIANT POINTS OR FIXED POINTS

If a point z maps ont itself i.e., w = z then the point is called invariant point or a fixed point of the bilinear transformation

PROPERTIES OF BLT

1. The cross ratio of a set of 4 points remain invariant under a BLT
$$\frac{(W_1 - W_2)(W_3 - W_4)}{(W_2 - W_3)(W_4 - W_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$
2. Every BLT map circles or straight lines in z-plane into circles or straight lines in w- plane

PROBLEMS:

1. **Find the BLT that maps the points (0,-i,-1) of z-plane onto the points (i,1,0) of w-plane respectively**.

Soln.

Given (z_1, z_2, z_3, z_4)= (z , 0. –i, 0)

(w_1, w_2, w_3, w_4)= (w, i, 1, 0)

Using cross ratio of 4 points

$$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\frac{w-i}{w} = \frac{(1-i)^2}{-i} \cdot \frac{z}{z+1}$$

$$\frac{w-i}{w} = 2 \cdot \frac{z}{z+1}$$

$$wz - iz + w - i = 2zw$$

$$w = \frac{iz+1}{-z+1}$$

This is the required BLT

2. **Find the BLT that maps the points (∞, i, 0) of z-plane onto the points (-1, -i, 1) of w-plane respectively.Also find the invariant points**.

Soln.

Given (z_1, z_2, z_3, z_4)= (z , ∞, i, 0)

(w_1, w_2, w_3, w_4)= (w,-1, -i, 1)

Using cross ratio of 4 points and $z_2 \to \infty \,\&\, 1/z_2 \to 0$

$$\frac{(w+1)(-1-i)}{(-1+i)(1`-w)} = \frac{(-1)(i)}{(1)(-z)}$$

$$\frac{(w+1)(1+i)}{(1-i)(1-w)} = \frac{i}{z}$$

$$zwi + zi + wz + z = i + 1 - wi - w$$

$$(w+1)(i+1)z = (i+1)(1-w)$$

$$w = \frac{-z+1}{z+1}$$

This is the required BLT

$$put \quad w = z$$

$$z = \frac{-z+1}{z+1}$$

$$z^2 + 2z - 1 = 0$$

$$z = -1 \pm \sqrt{2}$$

Are invariant points

COMPLEX INTEGRATION:

Consider xy-plane is taken as complex plane then the point p(x,y) on this curve corresponding to the complex number z=x-iy

The equation z=z(t) where t is parameter is reoffered to as the equation of curve in the complex form.

Ex: as t varies over the interval and x=a cast y=asin t then the complex form of the equation of circle is

$$z = x = iy$$

$$z = a\cos t = i\sin t$$

$$= a(cast + i\sin t$$

$$z = ae^{it} \qquad 0 \le t \le 2\pi$$

Represent a circle leaving center at origen. And radies is equal to a.

Consider a continuous function f^{z} of complex variable z=x=iy defined at all points of curve from p to Q

Divide the curve in to n equal parts by taking points p= $p(z_0)$, $p_1(z_1), p_2(z_2)$...

$p_{k-1}(Z_{k-1})$.......$p_n(z_n) = Q$ on the came C

The complex line integral along the path C is defined by $\int_c f(z)dz.$

If C is closed curve then $\oint_c f(z)dz.$

LINE INTEGRAL OF A COMPLEX VALUED FUNCTION

Let "D" is the region of complex plane and f(z)=u(x,y)+iv(x,y) be complex valued function defind on 'D' Let C be the curve in 'D' then (z=x+iy dz=dx+idy)

$$\int_c f(z)dz. = \int_c a(x,y) + iv(x,y)(dx = idy)$$

$$\int_c f(z)dz. = \int_c u(x,y)dx - v(x,y)dy + i\int_c vdx + udy$$

$$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$$

Is called the line integral of f(z) along the carve C

PROBLEMS:

1. Evaluate $\int_c z^2 dz$ Along the straight line from z=0 to z=3+i

Solution ; a) $\int_c z^2 dz = \int_{z=0}^{3+i} z^2 dz$

Here z is varies from o to 3+i

Z=x+iy

Z=0=) (0.0) (3,1)

Equation of the line gaining in $\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0}$

$$y = \frac{x}{3}$$

$$dx = 3.dy$$

$$f(z) = z^2 =)(x+iy)^2 = x^2 - y^2 + i2xy$$

$$\int_c z^2 dz = \int_{(0.0)}^{(3.1)} (x^2 - y^2)dx - 2xydy + i\int_{(0.0)}^{(3.1)} 2xydx + (x^2 - y^2)dy$$

Convert these integral in to ltu variable y or x respects to integral wi y from 0 to1 and x from 0to3

Use variable 'y'

$$\int_c z^2 dz = \int_0^1 \left[(9y^2 - y^2)3dy - 2(3y)ydy \right] + i\int_0^1 \left[2(3y)y3dy + i9y^2 - y^2)dy \right]$$

$$= \int_{y=0}^{1} (24y^2 - 6y^2)dy - + i\int_0^1 \left(8y^2 + 8y^2 \right) dy$$

$$i\int_0^1 18y^2 dy + i\int_0^1 26y^2 dy$$

$$= 18\left[\frac{y^2}{3}\right]_0^1 + 26i\left[\frac{y^3}{3}\right]_0^1 = 6 + \frac{26}{3}i$$

$$\int_c z^2 dz = 6 + \frac{26}{3}i$$

Along the given path.

2. Evaluate $\int_{(0.3)}^{(2,40} (2y = x^{2)dx=93x-y)dy}$ along the parabola x=2t, y=$t^2 = 3$

   X varies from 0to2 and here
   X=2t =) 0 =) 2t =) t=0
   X=2t =) 02=) 2t =) t=1
   T variesfrom 0to 1

$$\int_0^1 \left(2(t^2+3) = 4t^2 \right) 2.dt + \left(6t - (t^2+3) = 4t^2 \right) 2.t.dt$$

$$\int_0^1 (24t^2 - 2t^2 - 6t + 12)at$$

$$= 8t^3 - \frac{t^4}{2} - 3t^2 = 12t \int_0^1$$

$$=) 8 - \frac{1}{2} - 3 + 12 - (0)$$

$$=) \left(6 -^{1-} 6 + 24 \right) 2 =) \frac{33}{2}$$

Cauchy's Theorem:

Statement: If f(Z) is analytic at all points inside and on a simple Closed Curve C then

$$\int_c f(Z)dz = 0$$

Proof:

Let f(Z) dz = u+iv

$$\int f(z)dz = \int (u+iv)(dx+idy)$$

$$\int f(z)dz = \int (udx - vdy) + i\int (vdx + udy) \text{-------------------------- } 1$$

If M(x,y) and N(x,y) are two real Valued functioin then we have Green's Theorem

$$\int Mdx + Ndy = \iint\left(\frac{\partial n}{\partial x} - \frac{\partial m}{\partial y}\right)dxdy$$

Apply this theorem to 1

$$\int f(z)dz = \iint_R\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dxdy$$

By C R Equation

$$\int_c f(z)dz = 0$$

Corollary –II:

If c1 c2 are two simple closed Curves Such that c2 lies entirely with c1 and if f(z) is analytic on c1 c2 and in the region bounded by c1 c2 then $\int_{c1} f(z)dz = \int_{c2} f(z)dz$

Proof:

The Point p c1 and Q on c2 then the Curve PQRSTUQP as a simple closed curve on boundary C Hence by Cachy's theorem

$$\int f(z)dz = 0$$

Since C is the union of arc PRSP,PQ,QTUQ,AND QP THEN THEOREM BECOMES

$$\int_{c1} f(z)dz + \int_{c2} f(z)dz + \int_{-c3} f(z)dz - \int_{c4} f(z)dz = 0$$

Therefore

$$\int f(z)dz = \int_{c2} f(z)dz$$

Cauchy's Integral Formula:

Statement:

If f(z) is analytic inside and on a simple closed curve C and if a is any point with in C then

f(a) = $\frac{1}{2\pi i}\int\frac{f(z)}{z-a}dz$

Proof:

Since 'a' is a Point with in c we shall enclose it by a circle c1 with z-a at a centre and r as a radius such that c1 lies entirely with in c

The function f(z)/z-a is analytic inside and angular region between C and c1

By Cauchy's theorem

$$\int_c f(z)dz = \int_{c2} f(z)dz \ldots\ldots\ldots\ldots\ldots\ldots 1$$

Equation C1 can be written in the form i $z-a|$=r

Z-a = re$^{i\theta}$

dz= ire$^{i\theta}$dϴ

1 becomes

$$\int \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}.ire^{i\theta}d\theta$$

Hence is r is very small but r should greater than 0

$$\int \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a)d\theta$$

= if(a)[2a]

= 2πif(a)

There fore

f(a) = $\frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz$.

Problems :

1. Verify Cauchy's theorem for the function f(z) = z^2 where 'C'is squair leaving vertices (0,0) (1,0)91,1)(0,1)

$$\int_{oA} z^2dz + \int_{AB} z^2dz + \int_{BC} z^2dz + \int_{co} z^2dz = 0$$

$$z^2dz = (x+iy)^2(dx+idy)$$

$$x^2dx$$

$$\int_{oA} z^2dz = \int_{x=0} z^2dx =) \frac{x^3}{3}\int_0^1 =) \frac{1}{3}$$

$$z^2 =) (x+iy)^2(ddx+idy)$$

$$=) (1+i2y-y^2)idy$$

$$\int_{AB} z^2dz = i\int_{y=0}^{1}(1-y^2+i2y)dy$$

$$\int_{AB} z^2dz = \frac{2!}{3}-1$$

$$\int_{BC} z^2dz = \int_1^0 (x^2+2ix-1)dz$$

$$\int_{BC} z^2dz = \frac{2}{3}-i$$

$$\int_{CO} z^2dz = \int_1^0 (-y^2)idy$$

$$\int_{CO} z^2dz = \frac{i}{3}$$

Adding (1),(2) ,(3), (4)

$$\int_C z^2 dz = \frac{1}{3} + \frac{2i}{3} - 1 + \frac{2}{3} - i + \frac{i}{3}$$
$$= 0$$

2. Evaluate $\int_C \frac{c^z}{z+iy\Gamma}$ dz over each contour C $|z-1| = 1$

$$2\pi i = \int_c \frac{f(z)}{(z-a)} dz.$$
$$f(z) = e^z, \ |z-1| = 1$$

Soln: we have f(a) $a = 1, \ r = 1$

$$\int_c \frac{f(z)}{(z-a)} dz = 0$$
$$when \ |z-1| = 1$$

3. Evalute $\int_c \frac{z^2+z+1}{(z-2)^3} dz$ over c: $|z|$=3

Soln: The point z=2 lies inside circle

Causley's integral formula is

$$\int \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$
$$f(z) = z^2 + z + 1, \quad f'(z) = 2z + 1$$
$$f''(z) = 2$$
$$\int \frac{z^2+z+1}{(z-2)3} dz = \frac{2\pi i}{n!} f^2(2)$$
$$= \frac{2\pi i \times 2}{2!} 2\pi i$$

TAYLOR'S THEOREM:

If f(z) is analytic inside and on the boundary of the annular region 'R' bounded by two concentric circle C_1, C_2 with centered 'a' and radius r_1, r_2 then for every 'Z'

F(z) = $\sum_{n=0}^{\infty} an(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$

$$a_n = \frac{1}{2\pi i} \int_{c1} \frac{f(w)}{(w-a)^{n+1}} dw; a_{-n} = \frac{1}{2\pi i} \int_{c2} \frac{f(w)}{(w-a)^{n+1}} {}^{dw}$$

1. Show that $\frac{1}{z^2} = 1 + \sum_{k=1}^{\infty} (k+1)(z+1)^k \ where \ (z+1) < 1$

$$|z+1|<1$$

$$a=-1$$

$$f(z)=f(-1)+(z+1)f'(-1)+\frac{(z+1)^2}{2}f''(-1)+\frac{(z+1)^3}{\vartheta}\times f'''(-1)$$

$$f(z)=\frac{1}{z^2} \qquad f(-1)=1$$

$$f'(z)=\frac{2}{z^3} \qquad f'(-1)=2$$

$$f''(z)=\frac{6}{z^4} \qquad f''(-1)=6$$

$$f'''(z)=\frac{-24}{z^5} \qquad f'''(-1)=24$$

$$\frac{1}{z^2}=1+(z+1)2+\frac{(z+1)^2}{2}\times 6+\ldots\ldots\ldots$$

Soln: $1+\sum_{k=1}^{\infty}(k+1)(z+1)^k$

2. **Expand** $f(z)=\frac{1}{(z-1)(z-2)}$ **in terms of Laurent's series valid in the regions i)** $|z-1|<1$ **ii)** $|z-1|>1$**.**

Solution:

Given $f(z)=\frac{1}{(z-1)(z-2)}$ **is proper fraction**

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{2}{z-2}\ldots\ldots\ldots\ldots(1)$$

Case(i): If $|z-1|<1$**, put z-1 = u then** $\left|\frac{u}{1}\right|<1$

(1) Becomes $\therefore f(z)=\frac{-1}{(z-1)}+\frac{2}{z-1-1}$

$$=\frac{-1}{u}+\frac{2}{-(1-u)}$$

$$=\frac{-1}{u}+\left[-(1-u)^{-1}\right]=\frac{-1}{u}-2\left[1+u+u^2+\ldots\ldots\ldots\right]$$

$$=\frac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\ldots\ldots\ldots\right]$$

Case(ii): If $|z-1|>1$**, put z-1 = u**

$$i.e., |u|>1\Rightarrow\left|\frac{1}{u}\right|<1$$

(1) Becomes $\therefore\ f(z)=\dfrac{-1}{(z-1)}+\dfrac{2}{z-1-1}$

$$=\frac{-1}{u}+\frac{2}{u-1}=\frac{-1}{u}+\frac{2}{u(1-\frac{1}{u})}$$

$$=\frac{-1}{u}+\frac{2}{u}\left[(1-\frac{1}{u})^{-1}\right]=\frac{-1}{u}+\frac{2}{u}\left[1+\frac{1}{u}+\left(\frac{1}{u}\right)^{2}.........\right]$$

$$=\frac{-1}{z-1}+\frac{2}{u}\left[1+\frac{1}{z-1}+\left(\frac{1}{z-1}\right)^{2}..........\right]$$

3. Expand $f(z)=\dfrac{z+1}{(z+2)(z+3)}$ **in terms of Laurent's series valid in the regions i)** $|z|>3$ **ii)** $2<|z|<3$**.**

Solution:

We shall first resolve f(z) into partial fraction

i.e., $\therefore\ f(z)=\dfrac{-1}{(z+2)}+\dfrac{2}{z+3}............(1)$

case(i): $|z|>3$ **this implies that** $|z|>2$ **also.**

i.e., $\dfrac{|z|}{3}>1,\dfrac{|z|}{2}>1,$ **or** $\dfrac{3}{|z|}<1,\dfrac{2}{|z|}<1$

Hence we have to write (1) in the form

$$f(z)=\frac{-1}{z\left(1+\frac{2}{z}\right)}+\frac{2}{z\left(1+\frac{3}{z}\right)}$$

$$f(z)=\frac{1}{z}\left\{2\left(1+\frac{3}{z}\right)^{-1}-\left(1+\frac{2}{z}\right)^{-1}\right\}$$

$$f(z)=\frac{1}{z}\left\{2\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+.......\right)-\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+.......\right)\right\}$$

$$f(z)=\frac{1}{z}\left\{1-\frac{4}{z}+\frac{14}{z^2}-\frac{46}{z^3}+.......\right\}$$

Case(ii): $2<|z|<3$ **this implies that** $\dfrac{2}{|z|}<1,\dfrac{|z|}{3}<1$

Hence we have to write (1) in the form

$$f(z) = \frac{-1}{z\left(1+\frac{2}{z}\right)} + \frac{2}{3\left(1+\frac{z}{3}\right)}$$

$$f(z) = -\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{2}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = -\frac{1}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\ldots\ldots\right)+\frac{2}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\ldots\ldots\ldots\right)$$

$$Thus f(z) = -\frac{1}{z}+\frac{2}{z^2}-\frac{4}{z^3}+\frac{8}{z^4}+\ldots\ldots\ldots+\frac{2}{3}-\frac{2z}{3}+\frac{2z^2}{27}-\frac{2z^3}{81}+\ldots\ldots\ldots$$

4. **Find the Taylor's series expansion of** $f(z) = \frac{1}{(z+1)^2}$ **about the point z = -i.**

Solution:

For $f(z) = \frac{1}{(z+1)^2}$**, we have**

$$f(-i) = \frac{1}{(1-i)^2} = \frac{1}{-2i} = \frac{i}{2}$$

$$\& \; f^{(n)}(z) = \frac{(-1)^n (n+1)!}{(z+1)^{n+2}}$$

$$\therefore f^{(n)}(-i) = \frac{(-1)^n (n+1)!}{(-i+1)^{n+2}} = \frac{1}{(1-i)^2} \cdot \frac{(-1)^n (n+1)!}{(-i+1)^n} = \frac{i}{2}\frac{(-1)^n (n+1)!}{(-i+1)^n}$$

Therefore the Taylor's expansion of the given f(z) about the point z = -i is

$$f(z) = f(-i) + \sum_{n=1}^{\infty} \frac{f^{(n)}(-i)}{n!}(z+i)^n = \frac{i}{2}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n (n+1)!}{(-i+1)^n}(z+i)^n\right\}$$

5. **Find the Taylor's series expansion of** $f(z) = \frac{2z^3+1}{z^2+z}$ **about the point z = i.**

Solution:

For $f(z) = \frac{2z^3+1}{z^2+z} = (2z-1)+\frac{2z+1}{z^2+z} = 2(z-1)+\frac{(z+1)+z}{z(z+1)} = 2(z-1)+\frac{1}{z}+\frac{1}{z+1}$

This gives

$$f'(z) = 2-\frac{1}{z^2}+\frac{1}{(z+1)^2} \quad \textbf{and} \quad f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}+\frac{(-1)^n n!}{(z+1)^{n+1}}, n \geq 2$$

Accordingly,

$$f(i) = \frac{1-2i}{i(1+i)} = \frac{(-i)(1-2i)(1-i)}{(1+i)(1-i)} = -\frac{i}{2}(-1-3i) = \frac{i}{2}-\frac{3}{2}$$

$$f'(i) = 2-\frac{1}{i^2}-\frac{1}{(i+1)^2} = 2+1-\frac{1}{2i} = 3+\frac{i}{2}$$

And

$$f^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}} + \frac{(-1)^n n!}{(i+1)^{n+1}} = (-1)^n n!\left\{\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}}\right\}, n \geq 2$$

Hence, the Taylor's expansion of the given f(z) about the point z = i is

$$f(z) = f(i) + \sum_{n=1}^{\infty} \frac{f^{(n)}(i)}{n!}(z-i)^n = \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty}(-1)^n (n)!\left\{\frac{1}{i^{n+1}} + \frac{1}{(i+1)^{n+1}}\right\}z-i)^n$$

www.rejinpaul.com

www.rejinpaul.com

Unit-V
SPECIAL FUNCTIONS

Many Differential equations arising from physical problems are linear but have variable coefficients and do not permit a general analytical solution in terms of known functions. Such equations can be solved by numerical methods (Unit – I), but in many cases it is easier to find a solution in the form of an infinite convergent series. The series solution of certain differential equations gives rise to special functions such as Bessel's function, Legendre's polynomial. These special functions have many applications in engineering.

Series solution of the Bessel Differential Equation

Consider the Bessel Differential equation of order n in the form

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \text{(i)}$$

where n is a non negative real constant or parameter.

We assume the series solution of (i) in the form

$$y=\sum_{r=0}^{\infty}a_r x^{k+r} \quad \text{where } a_0\neq 0 \qquad \text{(ii)}$$

Hence,
$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}$$
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}$$

Substituting these in (i) we get,

$$x^2\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty}a_r(k+r)x^{k+r-1}+(x^2-n^2)\sum_{r=0}^{\infty}a_r x^{k+r}=0$$

i.e., $\sum_{r=0}^{\infty}a_r(k+r)(k+r-1)x^{k+r}+\sum_{r=0}^{\infty}a_r(k+r)x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}-n^2\sum_{r=0}^{\infty}a_r x^{k+r}=0$

Grouping the like powers, we get

$$\sum_{r=0}^{\infty}a_r\left[(k+r)(k+r-1)+(k+r)-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}=0$$

$$\sum_{r=0}^{\infty}a_r\left[(k+r)^2-n^2\right]x^{k+r}+\sum_{r=0}^{\infty}a_r x^{k+r+2}=0 \qquad \text{(iii)}$$

Now we shall equate the coefficient of various powers of x to zero

Equating the coefficient of x^k from the first term and equating it to zero, we get

$a_0\left[k^2 - n^2\right] = 0.$ Since $a_0 \neq 0$, we get $k^2 - n^2 = 0, \quad \therefore k = \pm n$

Coefficient of x^{k+1} is got by putting r = 1 in the first term and equating it to zero, we get

i.e., $a_1\left[(k+1)^2 - n^2\right] = 0.$ This gives $a_1 = 0$, since $(k+1)^2 - n^2 = 0$ gives, $k + 1 = \pm n$ which is a contradiction to $k = \pm n$.

Let us consider the coefficient of x^{k+r} from (iii) and equate it to zero.

i.e, $a_r\left[(k+r)^2 - n^2\right] + a_{r-2} = 0.$

$$\therefore a_r = \frac{-a_{r-2}}{\left[(k+r)^2 - n^2\right]} \qquad \text{(iv)}$$

If $k = +n$, (iv) becomes

$$a_r = \frac{-a_{r-2}}{\left[(n+r)^2 - n^2\right]} = \frac{-a_{r-2}}{\left[r^2 + 2nr\right]}$$

Now putting r = 1,3,5, ….., (odd vales of n) we obtain,

$$a_3 = \frac{-a_1}{6n+9} = 0, \quad \because a_1 = 0$$

Similarly a_5, a_7, ….. are equal to zero.

i.e., $a_1 = a_5 = a_7 = \ldots\ldots = 0$

Now, putting r = 2,4,6, ……(even values of n) we get,

$$a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}; \qquad a_4 = \frac{-a_2}{8n+16} = \frac{a_0}{32(n+1)(n+2)};$$

Similarly we can obtain a_6, a_8, …

We shall substitute the values of $a_1, a_2, a_3, a_4, \cdots\cdots$ in the assumed series solution, we get

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} = x^k(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\cdots)$$

Let y_1 be the solution for $k = +n$

$$\therefore \quad y_1 = x^n\left[a_0 - \frac{a_0}{4(n+1)}x^2 + \frac{a_0}{32(n+1)(n+2)}x^4 - \cdots\cdots\right]$$

$$i.e.,\quad y_1 = a_0 x^n \left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots\cdots\right] \qquad \text{(v)}$$

This is a solution of the Bessel's equation.

Let y_2 be the solution corresponding to k = - n. Replacing n be – n in (v) we get

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots\cdots\right] \qquad \text{(vi)}$$

The complete or general solution of the Bessel's differential equation is $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitrary constants.

Now we will proceed to find the solution in terms of Bessel's function by choosing $a_0 = \dfrac{1}{2^n \Gamma(n+1)}$ and let us denote it as Y_1.

$$i.e.,\quad Y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\cdot 2} - \cdots\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{(n+1)(n+2)\Gamma(n+1)\cdot 2} - \cdots\cdots\right]$$

We have the result Γ(n) = (n – 1) Γ(n – 1) from Gamma function

Hence, Γ(n + 2) = (n + 1) Γ(n + 1) and
Γ(n + 3) = (n + 2) Γ(n + 2) = (n + 2) (n + 1) Γ(n + 1)

Using the above results in Y_1, we get

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(n+2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(n+3)\cdot 2} - \cdots\cdots\right]$$

which can be further put in the following form

$$Y_1 = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0}{\Gamma(n+1)\cdot 0!}\left(\frac{x}{2}\right)^0 + \frac{(-1)^1}{\Gamma(n+2)\cdot 1!}\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{\Gamma(n+3)\cdot 2!}\left(\frac{x}{2}\right)^4 + \cdots\cdots\right]$$

$$= \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\cdot r!}\left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

This function is called the Bessel function of the first kind of order n and is denoted by $J_n(x)$.

Thus $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$

Further the particular solution for *k* = *-n* (replacing n by –n) be denoted as $J_{-n}(x)$. Hence the general solution of the Bessel's equation is given by $y = AJ_n(x) + BJ_{-n}(x)$, where *A* and *B* are arbitrary constants.

Properties of Bessel's function

1. $J_{-n}(x) = (-1)^n J_n(x)$, where n is a positive integer.

Proof: By definition of Bessel's function, we have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!} \qquad \text{……….(1)}$$

Hence, $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!}$ ……….(2)

But gamma function is defined only for a positive real number. Thus we write (2) in the following from

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\Gamma(-n+r+1) \cdot r!} \qquad \text{………..(3)}$$

Let r – n = s or r = s + n. Then (3) becomes

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{-n+2s+2n} \cdot \frac{1}{\Gamma(s+1) \cdot (s+n)!}$$

We know that $\Gamma(s+1) = s!$ and $(s + n)! = \Gamma(s+n+1)$

$$= \sum_{s=0}^{\infty} (-1)^{s+n} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

$$= (-1)^n \sum_{s=0}^{\infty} (-1)^{s} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{1}{\Gamma(s+n+1) \cdot s!}$$

Comparing the above summation with (1), we note that the RHS is $J_n(x)$.

Thus, $\boldsymbol{J_{-n}(x) = (-1)^n J_n(x)}$

2. $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$, *where n is a positive integer*

Proof: By definition, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$

$$\therefore \quad J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(-\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

i.e., $$= \sum_{r=0}^{\infty} (-1)^r \cdot (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

Thus, $\boldsymbol{J_n(-x) = (-1)^n J_n(x)}$

Since, $(-1)^n J_n(x) = J_{-n}(x)$, we have $\boldsymbol{J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)}$

Recurrence Relations:

Recurrence Relations are relations between Bessel's functions of different order.

Recurrence Relations 1: $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

From definition,

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!} = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2(n+r)} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)x^{2(n+r)-1}}{2^{n+2r}\Gamma(n+r+1)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+r)x^{n+2r-1}}{2^{n+2r-1}(n+r)\Gamma(n+r)\cdot r!}$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(x/2)^{(n-1)+2r}}{\Gamma(n-1+r+1)\cdot r!} = x^n J_{n-1}(x)$$

Thus, $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$ --------(1)

Recurrence Relations 2: $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

From definition,

$$x^{-n} J_n(x) = x^{-n} \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \cdot \frac{1}{\Gamma(n+r+1)\cdot r!}$$

$$\therefore \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r\,x^{2r-1}}{2^{n+2r}\Gamma(n+r+1)\cdot r!}$$

$$= -x^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\Gamma(n+r+1)\cdot (r-1)!}$$

Let $k = r - 1$

$$= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{n+1+2k}}{2^{n+1+2k} \Gamma(n+1+k+1) \cdot k!} = -x^{-n} J_{n+1}(x)$$

Thus, $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$ --------(2)

Recurrence Relations 3: $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

We know that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

Applying product rule on LHS, we get $x^n J_n^{/}(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

Dividing by x^n we get $J_n^{/}(x) + (n/x) J_n(x) = J_{n-1}(x)$ --------(3)

Also differentiating LHS of $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$, we get

$$x^{-n} J_n^{/}(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing by $-x^{-n}$ we get $-J_n^{/}(x) + (n/x) J_n(x) = J_{n+1}(x)$ --------(4)

Adding (3) and (4), we obtain $2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$

i.e., $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

Recurrence Relations 4: $J_n^{/}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Subtracting (4) from (3), we obtain $2J_n^{/}(x) = \left[J_{n-1}(x) - J_{n+1}(x)\right]$

i.e., $J_n^{/}(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Recurrence Relations 5: $J_n^{/}(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

This recurrence relation is another way of writing the Recurrence relation 2.

Recurrence Relations 6: $J_n^{/}(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

This recurrence relation is another way of writing the Recurrence relation 1.

Recurrence Relations 7: $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

This recurrence relation is another way of writing the Recurrence relation 3.

Problems:

Prove that $(a)\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad (b)\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

By definition,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Putting n = ½, we get

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{1/2+2r} \cdot \frac{1}{\Gamma(r+3/2) \cdot r!}$$

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(3/2)}-\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(5/2)1!}+\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(7/2)2!}-\cdots\cdots\right] \quad \text{--------(1)}$$

Using the results $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(n) = (n-1)\,\Gamma(n-1)$, we get

$\Gamma(3/2)=\frac{\sqrt{\pi}}{2}, \Gamma(5/2)=\frac{3\sqrt{\pi}}{4}, \Gamma(7/2)=\frac{15\sqrt{\pi}}{8}$ and so on.

Using these values in (1), we get

$$J_{1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}}-\frac{x^2}{4}\frac{4}{3\sqrt{\pi}}+\frac{x^4}{16}\frac{8}{15\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{x}{2\pi}}\cdot\frac{2}{x}\left[x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\cdots\right]=\sqrt{\frac{2}{x\pi}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots\right]$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

Putting n = - 1/2, we get

$$J_{-1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\cdot\left(\frac{x}{2}\right)^{-1/2+2r}\cdot\frac{1}{\Gamma(r+1/2)\cdot r!}$$

$$J_{-1/2}(x)=\sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma(1/2)}-\left(\frac{x}{2}\right)^2\frac{1}{\Gamma(3/2)1!}+\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)2!}-\cdots\cdots\right] \quad \text{--------(2)}$$

Using the results $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(n) = (n-1)\,\Gamma(n-1)$ in (2), we get

$$J_{-1/2}(x)=\sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}}-\frac{x^2}{4}\frac{2}{\sqrt{\pi}}+\frac{x^4}{16}\frac{4}{3\sqrt{\pi}.2}-\cdots\cdots\right]$$

$$=\sqrt{\frac{2}{x\pi}}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\cdots\right]$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

2. Prove the following results :

(a) $J_{5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x-\frac{3}{x}\cos x\right]$ and

(b) $J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x+\frac{3}{x}\sin x\right]$

Solution :

We prove this result using the recurrence relation $J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$ ------ (1).

Putting n = 3/2 in (1), we get $J_{1/2}(x)+J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)$

$\therefore\quad J_{5/2}(x)=\frac{3}{x}J_{3/2}(x)-J_{1/2}(x)$

i.e., $J_{5/2}(x)=\frac{3}{x}\sqrt{\frac{2}{\pi x}}\left[\frac{\sin x-x\cos x}{x}\right]-\sqrt{\frac{2}{\pi x}}\sin x$

$$J_{5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x-3x\cos x-x^2\sin x}{x^2}\right]=\sqrt{\frac{2}{\pi x}}\left[\frac{(3-x^2)}{x^2}\sin x-\frac{3}{x}\cos x\right]$$

Also putting n = - 3/2 in (1), we get $J_{-5/2}(x)+J_{-1/2}(x)=-\frac{3}{x}J_{-3/2}(x)$

$$\therefore \quad J_{-5/2}(x)=-\frac{3}{x}J_{-3/2}(x)-J_{-1/2}(x)=\left(\frac{-3}{x}\right)\left(-\sqrt{\frac{2}{\pi x}}\right)\left[\frac{x\sin x+\cos x}{x}\right]-\sqrt{\frac{2}{\pi x}}\cos x$$

$$i.e., \quad J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3x\sin x+3\cos x-x^2\cos x}{x^2}\right]=\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x+\frac{3-x^2}{x^2}\cos x\right]$$

3. Show that $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

Solution:

L.H.S = $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)J_n'(x)+2J_{n+1}(x)J_{n+1}'(x)$ ------- (1)

We know the recurrence relations

$$xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x) \qquad \text{------- (2)}$$

$$xJ_{n+1}'(x)=xJ_n(x)-(n+1)J_{n+1}(x) \qquad \text{------- (3)}$$

Relation (3) is obtained by replacing ***n*** by ***n+1*** in $xJ_n'(x)=xJ_{n-1}(x)-nJ_n(x)$

Now using (2) and (3) in (1), we get

L.H.S $=\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=2J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+2J_{n+1}(x)\left[J_n(x)-\frac{n+1}{x}J_{n+1}(x)\right]$

$$=\frac{2n}{x}J_n^2(x)-2J_n(x)J_{n+1}(x)+2J_{n+1}(x)J_n(x)-2\frac{n+1}{x}J_{n+1}^2(x)$$

Hence, $\frac{d}{dx}\left[J_n^2(x)+J_{n+1}^2(x)\right]=\frac{2}{x}\left[nJ_n^2(x)-(n+1)J_{n+1}^2(x)\right]$

4. Prove that $J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right]$

Solution :

We have the recurrence relation $J_n'(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$ -------(1)

Putting n = 0 in (1), we get $J_0'(x)=\frac{1}{2}\left[J_{-1}(x)-J_1(x)\right]=\frac{1}{2}\left[-J_1(x)-J_1(x)\right]=-J_1(x)$

Thus, $J_0'(x)=-J_1(x)$. Differentiating this w.r.t. x we get, $J_0''(x)=-J_1'(x)$ ----- (2)

Now, from (1), for n = 1, we get $J_1'(x)=\frac{1}{2}\left[J_0(x)-J_2(x)\right]$.

Using (2), the above equation becomes

$$-J_0''(x)=\frac{1}{2}\left[J_0(x)-J_2(x)\right] \text{ or } J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right].$$

Thus we have proved that, $J_0''(x)=\frac{1}{2}\left[J_2(x)-J_0(x)\right]$

5. Show that (a) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Solution :

(a) We know that $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$ or $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ------ (1)

Now, $\int J_3(x)dx = \int x^2 \cdot x^{-2}J_3(x)dx + c = x^2 \cdot \int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$

$= x^2 \cdot \left[-x^{-2}J_2(x)\right] - \int 2x\left[-x^{-2}J_2(x)\right]dx + c$ (from (1) when n = 2)

$= c - J_2(x) - \int \frac{2}{x}J_2(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$ (from (1) when n = 1)

Hence, $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$

(b) $\int xJ_0^2(x)dx = J_0^2(x) \cdot \frac{1}{2}x^2 - \int 2J_0(x) \cdot J_0'(x) . \frac{1}{2}x^2 dx$ (Integrate by parts)

$= \frac{1}{2}x^2 J_0^2(x) + \int x^2 J_0(x) \cdot J_1(x)dx$ (From (1) for n = 0)

$= \frac{1}{2}x^2 J_0^2(x) + \int xJ_1(x) \cdot \frac{d}{dx}\left[xJ_1(x)\right]dx \quad \left[\because \frac{d}{dx}\left[xJ_1(x)\right] = xJ_0(x) \text{ from recurrence relation (1)}\right]$

$= \frac{1}{2}x^2 J_0^2(x) + \frac{1}{2}\left[xJ_1(x)\right]^2 = \frac{1}{2}x^2\left[J_0^2(x) + J_1^2(x)\right]$

Generating Function for $J_n(x)$

To prove that $e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

or

If n is an integer then $J_n(x)$ is the coefficient of t^n in the expansion of $e^{\frac{x}{2}(t-1/t)}$.

Proof:

We have $e^{\frac{x}{2}(t-1/t)} = e^{xt/2} \times e^{-x/2t}$

$$= \left[1 + \frac{(xt/2)}{1!} + \frac{(xt/2)^2}{2!} + \frac{(xt/2)^3}{3!} + \cdots\cdots\right] \bullet \left[1 + \frac{(-xt/2)}{1!} + \frac{(-xt/2)^2}{2!} + \frac{(-xt/2)^3}{3!} + \cdots\cdots\right]$$

(using the expansion of exponential function)

$$= \left[1 + \frac{xt}{2 \cdot 1!} + \frac{x^2t^2}{2^2 2!} + \cdots + \frac{x^n t^n}{2^n n!} + \frac{x^{n+1}t^{n+1}}{2^{n+1}(n+1)!} + \cdots\cdots\right] \bullet \left[1 - \frac{x}{2t \cdot 1!} + \frac{x^2}{2^2 t^2 2!} - \cdots + \frac{(-1)^n x^n}{2^n t^n n!} + \frac{(-1)^{n+1}x^{n+1}}{2^{n+1}t^{n+1}(n+1)!} + \cdots\cdots\right]$$

If we collect the coefficient of t^n in the product, they are

$$= \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!1!} + \frac{x^{n+4}}{2^{n+4}(n+2)!2!} - \cdots\cdots$$

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!1!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4} - \cdots\cdots = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)r!} = J_n(x)$$

Similarly, if we collect the coefficients of t^{-n} in the product, we get $J_{-n}(x)$.

Thus, $e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

Result: $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty} \left[t^n + (-1)^n t^{-n}\right] J_n(x)$

Proof :

$$e^{\frac{x}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{-1} t^n J_n(x) + \sum_{n=0}^{\infty} t^n J_n(x)$$

$$= \sum_{n=1}^{\infty} t^{-n} J_{-n}(x) + J_0(x) + \sum_{n=1}^{\infty} t^n J_n(x) = J_0(x) + \sum_{n=1}^{\infty} t^{-n}(-1)^n J_n(x) + \sum_{n=1}^{\infty} t^n J_n(x) \quad \{\because J_{-n}(x) = (-1)^n J_n(x)\}$$

Thus, $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty} \left[t^n + (-1)^n t^{-n}\right] J_n(x)$

Problem 6: Show that

(a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$ *, n being an integer*

(b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Solution :

We know that $e^{\frac{x}{2}(t-1/t)} = J_0(x) + \sum_{n=1}^{\infty} \left[t^n + (-1)^n t^{-n}\right] J_n(x)$

$$= J_0(x) + tJ_1(x) + t^2 J_2(x) + t^3 J_3(x) + \cdots\cdots + t^{-1} J_{-1}(x) + t^{-2} J_{-2}(x) + t^{-3} J_{-3}(x) + \cdots\cdots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$, we have

$$e^{\frac{x}{2}(t-1/t)} = J_0(x) + J_1(x)\left(t - 1/t\right) + J_2(x)\left(t^2 + 1/t^2\right) + J_3(x)\left(t^3 - 1/t^3\right) + \cdots\cdots \quad \text{----- (1)}$$

Let t = cosθ + i sinθ so that t^p = cospθ + i sinpθ and $1/t^p$ = cospθ - i sinpθ.
From this we get, $t^p + 1/t^p$ = 2cospθ and $t^p - 1/t^p$ = 2i sinpθ
Using these results in (1), we get

$$e^{\frac{x}{2}(2i\sin\theta)} = e^{ix\sin\theta} = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] + 2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right] \quad \text{-----(2)}$$

Since $e^{ix\sin\theta}$ = *cos(xsinθ) + i sin(xsinθ)*, equating real and imaginary parts in (2) we get,

$$\cos(x\sin\theta) = J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] \quad \text{----- (3)}$$

$$\sin(x\sin\theta) = 2\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right] \quad \text{----- (4)}$$

These series are known as ***Jacobi Series***.

Now multiplying both sides of (3) by *cos nθ* and both sides of (4) by *sin nθ* and integrating each of the resulting expression between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\,d\theta = \begin{cases} J_n(x), & \text{n is even or zero} \\ 0, & \text{n is odd} \end{cases}$$

and $\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta d\theta = \begin{cases} 0, & \text{n is even} \\ J_n(x), & \text{n is odd} \end{cases}$

Here we used the standard result $\int_0^{\pi} \cos p\theta \cos q\theta d\theta = \int_0^{\pi} \sin p\theta \sin q\theta d\theta = \begin{cases} \frac{\pi}{2}, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases}$

From the above two expression, in general, if n is a positive integer, we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\right] d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$$

(b) Changing θ to (π/2) θ in (3), we get

$$\cos(x\cos\theta) = J_0(x) + 2\left[J_2(x)\cos(\pi - 2\theta) + J_4(x)\cos(\pi - 4\theta) + \cdots\right]$$

$$\cos(x\cos\theta) = J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots$$

Integrating the above equation w.r.t θ from 0 to π, we get

$$\int_0^{\pi} \cos(x\cos\theta)d\theta = \int_0^{\pi} \left[J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta - \cdots\right]$$

$$\int_0^{\pi} \cos(x\cos\theta)d\theta = \left| J_0(x)\cdot\theta - 2J_2(x)\frac{\sin 2\theta}{2} + 2J_4(x)\frac{\sin 4\theta}{4} - \cdots \right|_0^{\pi} = J_0(x)\cdot\pi$$

Thus, $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)d\theta$

(c) Squaring (3) and (4) and integrating w.r.t. *θ from 0 to π* and noting that *m* and *n* being integers

$$\int_0^{\pi} \cos^2(x\sin\theta)d\theta = \left[J_0(x)\right]^2\cdot\pi + 4\left[J_2(x)\right]^2\frac{\pi}{2} + 4\left[J_4(x)\right]^2\frac{\pi}{2} + \cdots$$

$$\int_0^{\pi} \sin^2(x\sin\theta)d\theta = 4\left[J_1(x)\right]^2\frac{\pi}{2} + 4\left[J_3(x)\right]^2\frac{\pi}{2} + \cdots$$

Adding, $\int_0^{\pi} d\theta = \pi = \pi\left[J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + J_3^2(x) + \cdots\cdots\right]$

Hence, $J_0^2 + 2J_1^2 + 2J_2^2 + J_3^2 + \cdots\cdots = 1$

Orthogonality of Bessel Functions

If α and β are the two distinct roots of $\boldsymbol{J_n(x) = 0}$, *then*

$$\int_0^{\pi} xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left[J_n'(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2, & \text{if } \alpha = \beta \end{cases}$$

Proof:

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad \text{-------- (1)}$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad \text{-------- (2)}$$

are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively.

Multiplying (1) by ***v/x*** and (2) by ***u/x*** and subtracting, we get

$x(u'' v - u v'') + (u' v - uv') + (\beta^2 - \alpha^2) xuv = 0$

or $\frac{d}{dx}\left[x(u'v - uv')\right] = (\alpha^2 - \beta^2) xuv$

Now integrating both sides from 0 to 1, we get

$$(\beta^2 - \alpha^2)\int_0^1 xuv\,dx = \left[x(u'v - uv')\right]_0^1 = (u'v - uv')_{x=1} \quad \text{------- (3)}$$

Since $u = J_n(\alpha x)$, $u' = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d}{d(\alpha x)}\left[J_n(\alpha x)\right]\cdot\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$

Similarly $v = J_n(\beta x)$ gives $v' = \frac{d}{dx}\left[J_n(\beta x)\right] = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad \text{------- (4)}$$

If α and β are the two distinct roots of $\mathbf{J_n(x) = 0}$, then $J_n(\alpha) = 0$ and $J_n(\beta) = 0$, and hence (4) reduces to $\int_0^\pi xJ_n(\alpha x)J_n(\beta x)dx = 0$.

This is known as Orthogonality relation of Bessel functions.

When $\beta = \alpha$, the RHS of (4) takes 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching to α. Then (4) gives

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

Applying L'Hospital rule, we get

$$\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{Lt}\frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}\left\{J_n'(\alpha)\right\}^2 \text{--------(5)}$$

We have the recurrence relation $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$.

$\therefore \quad J_n'(\alpha) = \frac{n}{\alpha}J_n(\alpha) - J_{n+1}(\alpha)$. Since $J_n(\alpha) = 0$, we have $J_n'(\alpha) = -J_{n+1}(\alpha)$

Thus, (5) becomes $\underset{\beta\to\alpha}{Lt}\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{1}{2}\left\{J_n'(\alpha)\right\}^2 = \frac{1}{2}\left\{J_{n+1}(\alpha)\right\}^2$

LEGENDRE'S POLYNOMIAL

If n is a positive even integer, $a_0u(x)$ reduces to a polynomial of degree n and if n is positive odd integer $a_1v(x)$ reduces to a polynomial of degree n. Otherwise these will give infinite series called Legendre functions of second kind.

Polyninomials u(x),v(x) contain alternate powers of x and a general form of the polynomial that represents either of them in descending powers of x and can be presented in the form

$$y = f(x) = a_n x^n + a_{n-2}x^{n-2} + a_{n-4}x^{n-4} + \ldots\ldots\ldots + F(x)\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\text{where } F(x) = \begin{cases} a_0 \text{ if } n \text{ is even} \\ a_1 x \text{ if } n \text{ is odd} \end{cases}$$

We note that a_r is the coefficient of x^r in the series solution of the differential equation and we have obtained

$$a_{r+2} = \frac{-\left[n(n+1) - r(r+1)\right]}{(r+2)(r+1)} a_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We plan to express a_{n-2}, a_{n-4}.....present in (1) in terms of a_n. Replacing r by (n-2) in (2) we obtain

$$a_n = \frac{-\left[n(n+1) - (n-2)(n-1)\right]}{n(n-1)} a_{n-2} ..$$

$$a_n = \frac{-(4n-2)}{n(n-1)} a_{n-2}$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

Again from (2) on replacing r by (n-4) we obtain

$$a_{n-2} = \frac{-\left[n(n+1) - (n-4)(n-3)\right]}{(n-2)(n-3)} a_{n-4}$$

$$a_{n-2} = \frac{-\left[8n-12\right]}{\left[(n-2)(n-3)\right]} a_{n-4}$$

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$a_{n-2} = \frac{-\left[8n-12\right]}{\left[(n-2)(n-3)\right]} a_{n-4}$$

$$a_{n-4} = \frac{-n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} a_n \; by \; using \; values \; of \; a_{n-2}$$

$Using \; these \; values \; in \; (1) \; we \; have$

$$y = f(x) = a_n\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} \ldots\ldots\ldots + G(x)\right]$$

$$where\, G(x) = \begin{cases} a_0 / a_n & if \; n \; is \; even \\ a_1 x / a_n & if \; n \; is \; odd \end{cases}$$

If the constant a_n is so choosen such that y-f(x) becomes 1 when x=1, the polynomials are called Legendre polynomials denoted by $P_n(x)$.

Let us choose

$$a_n = \frac{1.3.5\ldots.2n-1}{n!}, \; that \; is$$

$$P_n(x) = \frac{1.3.5\ldots.2n-1}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} \ldots\ldots\ldots\right] \ldots\ldots\ldots(3)$$

We obtain few Legendre polynomials by putting n=0,1,2,3,4,

$$P_0 = 1, \; P_1(x) = x, \; P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \; P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \; P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) etc\ldots.$$

Rodrigues' formula

WE derive a formula for the legendre polynomials $P_n(x)$ in the form

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \quad Known\ as\ Rodrigue's\ formula.$$

$$u = (x^2-1)^n$$

nth derivative is a solution of the Legendre's differential equation

Proof. Let $(1-x^2)y'' - 2xy' + n(n+1)y = 0 ....(1)$

differentiation u w.r.t

$$\frac{du}{dx} = u_1 = n(x^2-1)^{n-1}2x \ \ i.e\ (x^2-1)u_1 = 2nxu$$

Differentiating w.r.t x again we have

$$(x^2-1)u_2 + 2xu_1 = 2n(xu_1 + u)$$

Differentiate the result n times by applying Lebnitz theorem for the nth derivate

$$\left[(x^2-1)u_2\right]_n + 2\left[xu_1\right]_n = 2n\left[xu_1\right]_n + 2nu_1$$

$$\left[(x^2-1)u_{n+2}\right] + 2\left[xu_{n+1}\right] - n^2u_n - nu_n = 0$$

$$(1-x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0..........(2)$$

Comparing (2) with (1) we conclude that u_n is a solution of the legendre's differential equation.Also $P_n(x)$ which satisfies the legendre differential equation is also a polynomial of degree n . Hence u_n must be the same as $P_n(x)$ for some constant factor k

$$P_n(x) = ku_n = k\left[(x^2-1)^n\right]_n$$

$$P_n(x)_n = k\left[(x-1)^n(x+1)^n\right]_n \ \ apply\inf\ leibnitztheoremforRHS$$

$$P_n(x) = k\left[(x-1)^n\left\{(x+1)^n\right\}_n + n.n(x-1)^{n-1}\left\{(x+1)^n\right\}_{n-1} + \frac{n(n-1)}{2!}n(n-1)(x-1)^{n-2}\left\{(x+1)^n\right\}_{n-2} + ..\left\{(x-1)^n\right\}_n\right]..(3)$$

We proceed to find k by choosing a suitable value for x. putting x=1 in (3) in RHS becomes zero expect the last term

$$P_n(1) = k.n!.2^n \ \ or\ \ k = \frac{1}{n!2^n}$$

$$P_n(x) = ku_n,\ \ where\ \ P_n(x) = \frac{1}{n!2^n}\left\{(x^2-1)^n\right\}_n$$

$$Thus\ \ proved\ \ that\ \ P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}\left\{(x^2-1)^n\right\}....(Rodrigue's Formula.)$$

PROBLEMS:

$$1. If\ x^3+2x^2-x+1=aP_0(x)+bP_1(x)+cP_2(x)+dP_3(x)$$

Find the values of a,b,c,d

soln:

Let $f(x)=x^3+2x^2-x+1$

substitutuing for $x^3,x^2,x,1$ *in terms of legendre polynomial we have*

$$f(x)=\frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-\frac{2}{5}P_1(x)+\frac{5}{3}P_0(x)$$

hence we have

$$aP_0(x)+bP_1(x)+cP_2(x)+dP_3(x)$$

$$=\frac{5}{3}P_0(x)-\frac{2}{5}P_1(x)-\frac{4}{3}P_2(x)+\frac{2}{5}P_3(x)$$

thus by comparing both sides we obtain

$$a=\frac{5}{3},b=-\frac{2}{5},c=\frac{4}{3},d=\frac{2}{5}$$

$$2\ Show\ that\ x^4-3x^2+x=\frac{8}{35}P_4(x)-\frac{10}{7}P_2(x)+P_1(x)-\frac{4}{5}P_0(x)$$

soln:

Let $f(x)=x^4-3x^2+x$ *and we have obtained*

$$x^4=\frac{8}{35}P_4(x)+\frac{4}{7}P_2(x)+\frac{1}{5}P_0(x)$$

$$x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$$

substituting these in $f(x)$ *with* $x=P_1(x)$ *we have*

$$f(x)=\left(\frac{8}{35}P_4(x)+\frac{4}{7}P_2(x)+\frac{1}{5}P_0(x)\right)-3\left(\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right)+P_1(x)$$

thus

$$f(x)=\frac{8}{35}P_4(x)-\frac{10}{7}P_2(x)+P_1(x)-\frac{4}{5}P_0(x)$$

4. Obtain $P_3(x)$ from Rodrigue's formula and verify that the same satisfies the Legendre's equation in the standard form.

soln :

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

we have Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

We have to verify that

$$(1-x^2)P_3''(x) - 2xP'_3(x) + 3(3+1)P_3(x) = 0, \sin ce\ n = 3$$

From the exp*ression of* $P_3(x)$ *we get*

$$P'_3(x) = \frac{1}{2}(15x^2 - 3)\ and\ P_3''(x) = 15x$$

$$(1-x^2)P_3''(x) - 2xP'_3(x) + 12P_3(x)$$

$$= (1-x^2)15x - 2x\frac{1}{2}(15x^2 - 3) + 12\frac{1}{2}(5x^3 - 3x)$$

$$= 15x - 15x^3 - 15x^3 + 3x + 30x^3 - 18x = 0$$

Thus we have verified $P_3(x)$ *satifies Legendre's equation*

3. *show that* $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$

soln :

We have $P_2(x) = \frac{1}{2}(3x^2 - 1)$

now $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

$$P_2(\cos\theta) = \frac{1}{4}\ 3 + 3\cos 2\theta - 2$$

thus

$$P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$$

Unit VI
Probability Theory I

PROBABILITY:

Random experiment:

It is an experiment which performed repeatedly outcomes a result or an experiment performed repeatedly giving different results on outcomes are called random experiment.
Eg: tossing a coin, throwing a die.

Sample space:

Sample space of random experiment set of all possible outcomes & it is denoted by 'S' . The number of elements in the set is denoted by 0(S).

Event : event is a subset of sample space & is denoted by E
Exhaustive event: The set of events is said to be exhaustive if it includes all possible events.

Mutually exclusive events: Two events A & B are mutually exclusive, if A & B cannot happen simultaneously $\Rightarrow A \cap B = \phi$
A & B are disjoint $i.e. p \;\; A \cap B \;= 0$

Mutually independent events: Two events A & B are mutually independent if the occurrence of the event A does not depend on the occurrence of the event B.

Probability: If an event A can happen in M ways out of the possible n-ways (mutually exclusive & equally likely) then probability of A is denoted by P(A)

$$P(A) = \frac{m}{n} = \frac{\text{favourable number of cases}}{\text{Total number of cases}}$$

The probability of non-occurance of event A (A will not happen) is given by $P(\bar{A})$ or $P(A')$ or q

$$q = P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$\mathrm{P}(\bar{\mathrm{A}}) = 1 - P(A)$$

$$P(A) + \mathrm{P}(\bar{\mathrm{A}}) = 1$$

Axioms of probability:
i) for an event A of S, probability lies
ii) between $0 \le P(A) \le 1$ The numerical value of probability lies between 0 & 1, P(S)=1
iii) $P(A \cup B) = P(A) + P(B)$; A & B are disjoint

Addition theorem or rule of total probability
If A & B are any two elements, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Sol:

$A \cup B = A \cup (B \cap \bar{A})$

$P(A \cup B) = P(A \cup (B \cap \bar{A}))$ $\left[\text{A and } B \cap \bar{A} \text{ are disjoint}\right]$

$= P(A) + P(B \cap \bar{A})$

Add & subtract $P(A \cap B)$

$P(A \cup B) = P(A) + P(B \cap \bar{A}) + P(A \cap B) - P(A \cap B)$

$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

similarly,

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

If A, B, C are mutually exclusive, then

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Conditional probability: let A & B are two events, probability of the happening of event B when the event A has already occurred is called Conditional probability & is denoted by $P\left(B/A\right)$

$$P\left(B/A\right) = \frac{\text{Probability of occurance of both A \& B}}{\text{Probability of occurance of given event A}}$$

$\boxed{P\left(B/A\right) = \frac{P(A \cap B)}{P(A)}}$ $\rightarrow$ Multiplication rule of probability

$P(A \cap B) = P\left(B/A\right) . P(A)$

If $A \& B$ are Mutually independent event then, $P\left(B/A\right) = P(B)$

$\boxed{P(A \cap B) = P(B) . P(A)}$

Problems:

1. A boy & girl appeared in interview for 2 vacancy's in the same post. The probability of boy & girl selection is $1/7$ & that of girl selection is $1/5$,what is the probability that

(i) both will be selected
(ii) None of them will be selected
(iii) one of them will be selected
(iv) atleast one of them will be selected.

Sol: let A & B be the events of selection of boy girl respectively

www.rejinpaul.com

$P(A)=\frac{1}{7}, \quad P(B)=\frac{1}{5}$

Total space $=\frac{1}{7}+\frac{1}{5}=\frac{12}{35}$

i) Probability of both of them getting selected

(i) $P(A\cap B)=P(A).P(B)$

$$=\frac{1}{7}.\frac{1}{5}=\frac{1}{35}$$

(ii) probability of none of them getting selected

(ii) $P(\bar{A}\cap\bar{B})=P(\bar{A}).P(\bar{B})$

$$=\left[1-P(A)\right]\left[1-P(B)\right]$$

$$=\left(1-\frac{1}{7}\right)\left(1-\frac{1}{5}\right)$$

$$=0.685$$

(iii) Probability of one of them will be selected is

$$P(A\cap\bar{B})=P(A).P(\bar{B})+P(B).P(\bar{A})$$

$$=\left(\frac{1}{7}\right)\left(\frac{4}{5}\right)+\left(\frac{1}{5}\right)\left(\frac{6}{7}\right)$$

$$=\frac{4}{35}+\frac{6}{35}=0.285$$

(iv)probability of atleast one of them will be selected

$$=1-P(\bar{A}\cap\bar{B})$$

$$=1-0.685$$

$$=0.315$$

Probability that a cricket team wins a match is $3/5$. If the team wins 3 matches in the tournament what is the probability

i) team wins all matches ii) looses all match iii) wins atleast one match iv) win atmost one.

Sol:

$P(C) = \frac{3}{5}$

i)Probability that team wins all matches

$P(W_1) = P(W_2) = P(W_3) = 3/5$

$$= \left(\frac{3}{5}\right).\left(\frac{3}{5}\right).\left(\frac{3}{5}\right) = \frac{27}{125} = 0.216$$

(ii)probability of team loosing all matches

$$= \left(1-\frac{3}{5}\right)\left(1-\frac{3}{5}\right)\left(1-\frac{3}{5}\right)$$

$$= \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{8}{125} = 0.064$$

iii) probability that it wins atleast one match

$=1-0.064$

$=0.936$

iv) $= P(W_1)P(\bar{W_2})P(\bar{W_3}) + P(\bar{W_1})P(W_2)P(\bar{W_3}) + P(\bar{W_1})P(\bar{W_2})P(W_3)$

$$= \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)$$

$$=3\times\frac{12}{125} = \frac{36}{125} = 0.2888$$

BAYES THEORM:

Statement:

If $A_1, A_2, ...A_n$ are partition of a set 's' so that there union in s & E be any other event, then

$$\boxed{P\left(\frac{A_i}{E}\right) = \frac{P\left(\frac{E}{A_i}\right)P(A_i)}{\sum_{i=1}^{n} P\left(\frac{E}{A_i}\right)P(A_i)}}$$

Proof:

$S = A_1 \cup A_2 \cup A_3 \cup ....\cup A_n$

$E = E \cap S$

$E = E \cap (A_1 \cup A_2 \cup A_3 \cup ....\cup A_n)$

$\quad = (E \cap A_1) \cup (E \cap A_2) \cup ....\cup (E \cap A_n)$

$P(E) = P\left[(E \cap A_1) \cup (E \cap A_2) \cup ....\cup (E \cap A_n)\right]$

since $A_1, A_2, A_3, ...A_n$ are mutually disjoint set

$P(E) = P(E \cap A_1) + P(E \cap A_2) + ...+ P(E \cap A_n)$

$$\boxed{P(E) = \sum_{i=1}^{n} P(E \cap A_i)} \qquad (1)$$

By definition of conditional probability

$$P\left(\frac{A_i}{E}\right) = \frac{P(A_i \cap E)}{P(E)}$$

$$= \frac{P(E \cap A_i)}{P(E)}$$

$$= \frac{P(E \cap A_i)}{\sum_{i=1}^{n} P(E \cap A_i)} \text{ from (1)}$$

$$P\left(\frac{A_i}{E}\right) = \frac{P\left(\frac{E}{A_i}\right) P(A_i)}{\sum_{i=1}^{n} P(E \cap A_i)}$$

This result is known as theorm of inverse probability or Baye's theorm.

Problem;

1. In a college boys & girls are equal in proportion. It was found that 10 out of 100 boys & 25 out of 100 girls were using same company of 2 wheeles If the student using that was selected at random. What is the probability of being a boy.

Sol:

$$P\left(\frac{E}{A}\right) = \frac{10}{100} = 0.1; \; P(A) = 100\% = 0.5$$

$$P\left(\frac{E}{B}\right) = \frac{25}{100} = 0.25; \; P(A) = 100\% = 0.5$$

Let E be an event of choosing a student from same company of two wheelers

$$P\ \ E\ = P\ \ A\ .P\left(\frac{E}{A}\right) + P\ \ B\ .P\left(\frac{E}{B}\right)$$

$$= \ 0.5 \quad 0.1 \ + \ 0.5 \quad 0.25$$

$$\boxed{P\ \ E\ = 0.35}$$

probability of choosing a student from boy

$$P\left(\frac{A}{E}\right) = \frac{P\ \ A\ .P\left(\frac{E}{A}\right)}{P\ \ E}$$

$$= \frac{1 \quad 0.1}{0.35}$$

$$= 0.2857$$

2. In a school 25% of the students failed in first language, 15% of the students failed in second language and 10% of the students failed in both. If a student is selected at random find the probability that

(i) He failed in first language if he had failed in the second language.

(ii) He failed in second language if he had failed in the first language.

(iii) He failed in either of the two languages.

Solution: Let A be set of students failing in the first language and B be the set of students failing in the second language./ We have by data

$$P(A) = \frac{25}{100} = \frac{1}{4}, \quad P(B) = \frac{15}{100} = \frac{3}{20}, \quad P(A \cap B) = \frac{10}{100} = \frac{1}{10}$$

(i) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/10}{3/20} = \frac{2}{3}$

(ii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/10}{1/4} = \frac{2}{5}$

(iii) $P(AUB) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{3}{20} = \frac{1}{10} = \frac{3}{10}$

3. The probability that a team wins a match is 3/5. If this team play 3 matches in a tournament, what is the probability that the team

(i) Win all the matches

(ii) Win atleast one match

(iii) Win atmost one match

(iv) Lose all the matches

Solution: Let W be the event of winning a match by the team.

$P(W_1) = P(W_2) = P(W_3) = 3/5$

Let L be the event of losing a match by the team.

Therefore, $P(L_1) = P(L_2) = P(L_3) = 2/5$

www.rejinpaul.com

(I) Probability of winning all the matches

$= P(W_1)\ P(W_2)\ P(W_3) = 27/125$

ii) Probability of winning atleast one match

$= 1$ - Probability of lising all the matches

$= 1 - P(L_1)\ P(L_2)\ P(L_3)$

$= 1-(8/25) = 17/25$

iii) Probability of winning atmost one match.

$= P(L_1)\ P(L_2)\ P(L_3) + P(W_1)\ P(L_2)\ P(L_3) + P(L_1)\ P(W_2)\ P(L_3) + P(L_1)\ P(L_2)\ P(W_3)$

$= \frac{8}{125} + 3\left[\frac{3}{5}.\frac{2}{5.}.\frac{2}{5}\right] = \frac{44}{125}$

IV) Probability of losing all the matches

$= P(L_1)\ P(L_2)\ P(L_3) = \frac{8}{125}$

4. The odds that a book will reviewed favourably by 3 independent critics are 5 to 2, 4 to 3 and 3 to 4. Find the probability that majority of the reviews will be favourable.

Solution: Let E_1, E_2, E_3 be the events of favourable review by the three critics respectively.

Therefore $P(E1) = 5/7$, $P(E_2)=4/7$, $P(E_3) = 3/7$

Hence $P(\overline{E1}) = 2/7$, $P(\overline{E2})=3/7$, $P(\overline{E3}) = 4/7$

Majority of the reviews are favourable means that atleast two of three reviews should be favourable and if E denotes this event then we have

$P(E) = P(E1)\ P(E_2)\ P(\overline{E3}) + P(\overline{E1})\ P(E_2)\ P(E_3) + P(E1)\ P(\overline{E2})\ P(E_3) + P(E1)\ P(E_2)\ P(E_3)$

$= \frac{5}{7}.\frac{4}{7}.\frac{4}{7} + \frac{4}{7}.\frac{3}{7}.\frac{2}{7} + \frac{3}{7}.\frac{5}{7}.\frac{3}{7} + \frac{5}{7}.\frac{4}{7}.\frac{3}{7} = \frac{209}{343}$

5. **Three machines A, B, C produces 50%, 30% and 20% of the items in factory. The percentage of defective outputs are 3, 4, 5. If an item is selected at random. What is the probability that it is defective? What is the probability that it is from A?**

Solution: Let D denote the event of selecting of defective item

Given $P(A) = 0.5$ and $P(D/A) = 0.03$

$P(B) = 0.5$ and $P(D/B) = 0.04$

$P(C) = 0.5$ and $P(D/C) = 0.05$

Now $P(D) = P(A)\ P(D/A) + P(B)\ P(D/B) + P(C)\ P(D/C) = 0.037$

By Baye's theorem,

Probability that the defective item is from $A = P(A/D) = \frac{P(A)P(D/A)}{P(D)}$

$= \frac{(0.5)(0.03)}{0.037} = 0.4054$

6. In a college where boys and girls are equal proportion, it was found that 10 out of 100 boy and 25 out of 100 girls were using the same brand of a two wheeler. If a student using that was selected at random what is the probability of being a boy?

Solution: P(Boy) = P(B) = ½ = P(Girl)= P(G)

Let E be the event of choosing a student using that brand of vehicle.

Therefore, P(E/B) = 10/100 = 0.1 AND P(E/G) 25/100 = 0.25

Now, P(E) = P(B) P(E/B)+ P(G) P(E/G) = 0.175

We have to find P(B/E) AND BY Baye's theorem

$$P(B/E) = \frac{P(B)P(E/B)}{P(E)} = \frac{(0.5)(0.1)}{0.175} = 0.2857$$

Unit VII
Probability Theory II

In a practical situation, one may be interested in finding the probabilities of all the events and may wishes to have the results in a tabular form for any future reference. Since for an experiment having **n** outcomes, totally, there are $\mathbf{2^n}$ totally events; finding probabilities of each of these and keeping them in a tabular form may be an interesting problem.

Thus, if we develop a procedure, using which if it is possible to compute the probability of all the events, is certainly an improvement. The aim of this chapter is to initiate a discussion on the above.

Also, in many random experiments, outcomes may not involve a numerical value. In such a situation, to employ mathematical treatment, there is a need to bring in numbers into the problem. Further, probability theory must be supported and supplemented by other concepts to make application oriented. In many problems, we usually do not show interest on finding the chance of occurrence of an event, but, rather we work on an experiment with lot of expectations

Considering these in view, the present chapter is dedicated to a discussion of random variables which will address these problems.

<u>First what is a random variable?</u>

Let S denote the sample space of a random experiment. A random variable means it is a rule which assigns a numerical value to each and every outcome of the experiment. Thus, random variable may be viewed as a function from the sample space S to the set of all real numbers; denoted as $f : S \to R$. For example, consider the random experiment of tossing three fair coins up. Then S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}. Define f as the number of heads that appear. Hence, $f(HHH) = 3$, $f(HHT) = 2$, $f(HTH) = 2$, $f(THH) = 2$, $f(HTT) = 1$, $f(THT) = 1$, $f(TTH) = 1$ and $f(TTT) = 0$. The same can be explained by means of a table as given below:

www.rejinpaul.com

HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
3	2	2	2	1	1	1	1

Note that all the outcomes of the experiment are associated with a unique number. Therefore, f is an example of a random variable. Usually a random variable is denoted by using upper case letters such as X, Y, Z etc. The image set of the random variable may be written as $f(S) = \{0, 1, 2, 3\}$.

A random variable is divided into

- **Discrete Random Variable (DRV)**
- **Continuous Random Variable (CRV)**.

If the image set, **X(S),** is either finite or countable, then X is called as a discrete random variable, otherwise, it is referred to as a continuous random variable i.e. if X is a CRV, then X(S) is infinite and un – countable.

Example of Discrete Random Variables:

1. In the experiment of throwing a die, define X as the number that is obtained. Then X takes any of the values 1 – 6. Thus, X(S) = {1, 2, 3. . . 6} which is a finite set and hence X is a DRV.
2. Let X denotes the number of attempts required for an engineering graduate to obtain a satisfactory job in a firm? Then X(S) = {1, 2, 3,. . . . }. Clearly X is a DRV but having a image set countably infinite.
3. (iii) If X denote the random variable equals to the number of marks scored by a student in a subject of an examination, then X(S) = {0, 1, 2, 3, ... 100}. Thus, X is a DRV, Discrete Random Variable.
4. (iv) In an experiment, if the results turned to be a subset of the non – zero integers, Then it may be treated as a Discrete Random Variable.

Examples of Continuous Random Variable:

1. Let X denote the random variable equals the speed of a moving car, say, from a destination A to another location B, then it is known that speedometer indicates the speed of the car continuously over a range from 0 up to 160 KM per hour. Therefore, X is a CRV, Continuously Varying Random Variable.
2. Let X denotes the monitoring index of a patient admitted in ICU in a good hospital. Then it is a known fact that patient's condition will be watched by the doctors continuously over a range of time. Thus, X is a CRV.
3. Let X denote the number of minutes a person has to wait at a bus stop in Bangalore to catch a bus, then it is true that the person has to wait anywhere from 0 up to 20 minutes (say). Will you agree with me? Since waiting to be done continuously, random variable in this case is called as CRV.
4. Results of any experiments accompanied by continuous changes at random over a range of values may be classified as a continuous random variable.

Probability function/probability mass function $f(x_i) = P(X = x_i)$ of a discrete random variable:

Let X be a random variable taking the values, say $X: x_1 \quad x_2 \quad x_3 \; . \; . \; . \; x_n$ then $f(x_i) = P(X = x_i)$ is called as probability mass function or just probability function of the discrete random variable, X. Usually, this is described in a tabular form:

$X = x_i$	x_1	x_2	x_3	.	.	.	x_n
$f(x_i)$	$P(2 \le X < 5)\ f(x_1)$	$f(x_2)$	$f(x_3)$	.	.	.	$f(x_n)$

Note: When X is a discrete random variable, it is necessary to compute $f(x_i) = P(X = x_i)$ for each i = 1, 2, 3 . . n. This function has the following properties:

- $f(x_i) \ge 0$
- $0 \le f(x_i) \le 1$

- $\sum_i f\left(x_i\right) = 1$

On the other hand, X is a continuous random variable, then its probability function will be usually given or has a closed form, given as $f(x) = P(X = x)$ where x is defined over a range of values., it is called as probability density function usually has some standard form. This function too has the following properties:

- $f(x) \geq 0$
- $0 \leq f(x) \leq 1$
- $\int_{-\infty}^{\infty} f(x) = 1$.

To begin with we shall discuss in detail, discrete random variables and its distribution functions. Consider a discrete random variable, **X** with the distribution function as given below:

$X = x_i$	x_1	x_2	x_3	.	.	.	x_n
$f\left(x_i\right)$	$f\left(x_1\right)$	$f\left(x_2\right)$	$f\left(x_3\right)$	.	.	.	$f\left(x_n\right)$

Using this table, one can find probability of various events associated with X. For example,

- $P\left(x_i \leq X \leq x_j\right) = P\left(X = x_i\right) + P\left(X = x_{i+1}\right) + \text{ up to } + P\left(X = x_j\right)$

$$= f\left(x_i\right) + f\left(x_{i+1}\right) + f\left(x_{i+2}\right) + \text{up to } + f\left(x_{j-1}\right) + f\left(x_j\right)$$

- $P\left(x_i < X < x_j\right) = P\left(X = x_{i+1}\right) + P\left(X = x_{i+2}\right) + . \ . \ . + P\left(X = x_{j-1}\right)$

$$= f\left(x_{i+1}\right) + f\left(x_{i+2}\right) + \text{up to } + f\left(x_{j-1}\right)$$

- $P\left(X > x_j\right) = 1 - P\left(X \leq x_{j-1}\right) \quad = 1 - \left[P\left(X = x_1\right) + P\left(X = x_2\right) \text{ up to } + P\left(X = x_{j\text{-}1}\right)\right]$

The probability distribution function or cumulative distribution function is given as

$$F(x_t) = P(X \leq x_t) = P\left(X = x_1\right) + P\left(X = x_2\right) + \text{ up to } + P\left(X = x_t\right)$$

It has the following properties:

- $F(x) \geq 0$
- $0 \leq F(x) \leq 1$

- When $x_i < x_j$ then $F\left(x_i\right) < F\left(x_j\right)$ i.e. it is a strictly monotonic increasing function.
- when $x \to \infty$, $F(x)$ approaches 1
- when $x \to -\infty$, $F(x)$ approaches 0

<u>A brief note on Expectation, Variance, Standard Deviation of a Discrete Random Variable:</u>

- $E(X) = \sum_{i=1}^{i=n} x_i \cdot f\left(x_i\right)$
- $E\left(X^2\right) = \sum_{i=1}^{i=n} x_i^2 \cdot f\left(x_i\right)$
- $\mathrm{Var}(X) = E\left(X^2\right) - \left[E(X)\right]^2$

<u>ILLUSTRATIVE EXAMPLES:</u>

1. The probability density function of a discrete random variable X is given below:

X:	0	1	2	3	4	5	6
$f\left(x_i\right)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k; (ii) $F(4)$; (iii) $P(X \geq 5)$; (iv) $P(2 \leq X < 5)$ (v) E(X) and (vi) Var (X).

Solution: To find the value of k, consider the sum of all the probabilities which equals to 49k. Equating this to 1, we obtain $k = \frac{1}{49}$. Therefore, distribution of X may now be written as

X:	0	1	2	3	4	5	6
$f\left(x_i\right)$:	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Using this, we may solve the other problems in hand.

$$F(4) = P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4] = \frac{25}{49}.$$

$$P[X \geq 5] = P[X = 5] + P[X = 6] = \frac{24}{49}$$

$P[2 \le X < 5] = P[X=2] + P[X=3] + P[X=4] = \frac{21}{49}$. Next to find E(X), consider

$E(X) = \sum_i x_i \cdot f(x_i) = \frac{203}{49}$. To obtain Variance, it is necessary to compute

$E(X^2) = \sum_i x_i^2 \cdot f(x_i) = \frac{973}{49}$. Thus, Variance of X is obtained by using the relation, $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{973}{49} - \left(\frac{203}{49}\right)^2$.

2. A random variable, X, has the following distribution function.

X :	-2	-1	0	1	2	3
$f(x_i)$:	0.1	k	0.2	$2k$	0.3	k

Find (i) k, (ii) $F(2)$, (iii) $P(-2 < X < 2)$, (iv) $P(-1 < X \le 2)$, (v) E(X), Variance.

Solution: Consider the result, namely, sum of all the probabilities equals 1,

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$ Yields k = 0.1. In view of this, distribution function of X may be formulated as

X :	-2	-1	0	1	2	3
$f(x_i)$:	0.1	0.1	0.2	0.2	0.3	0.1

Note that $F(2) = P[X \le 2] = P[X=-2] + P[X=-1] + P[X=0] + P[X=1] + P[X=2]$

$= 0.9$. The same also be obtained using the result,

$F(2) = P[X \le 2] = 1 - P[X < 1] = 1 - \{P[X=-2] + P[X=-1] + P[X=0]\}$ **= 0.6.**

Next, $P(-2 < X < 2) = P[X=-1] + P[X=0] + P[X=1] = 0.5$.

Clearly, $P(-1 < X \le 2)$ **= 0.7. Now, consider** $E(X) = \sum_i x_i \cdot f(x_i)$ **= 0.8.**

Then $E(X^2) = \sum_i x_i^2 \cdot f(x_i)$ **= 2.8.** $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$ **= 2.8 - 0.64 = 2.16.**

<u>A DISCUSSION ON A CONTINUOUS RANDOM VARIABLE AND IT'S DENSITY FUNCTION:</u>

Consider a continuous random variable, X. Then its probability density is usually given in the form of a function $f(x)$ with the following properties.

(i) $f(x) \ge 0$, (ii) $0 \le f(x) \le 1$ and (*iii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Using the definition of $f(x)$, it is possible to compute the probabilities of various events associated with **X**.

- $P(a \le X \le b) = \int_a^b f(x)\,dx$, $\quad P(a < X \le b) = \int_a^b f(x)\,dx$
- $P(a < X < b) = \int_a^b f(x)\,dx$, $\quad F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$
- $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$, $\quad E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$
- $\mathrm{Var}(X) = E\left(X^2\right) - \left(E(X)\right)^2$
- $P(a < X < b) = F(b) - F(a)$
- $f(x) = \dfrac{dF(x)}{dx}$, **if the derivative exists**

SOME STANDARD DISTRIBUTIONS OF A DISCRETE RANDOM VARIABLE:

Binomial distribution function: Consider a random experiment having only two outcomes, say success (**S**) and failure (**F**). Suppose that trial is conducted, say, n number of times. One might be interested in knowing how many number of times success was achieved. Let **p** denotes the probability of obtaining a success in a single trial and **q** stands for the chance of getting a failure in one attempt implying that **p + q = 1**. If the experiment has the following characteristics;

- the probability of obtaining failure or success is same for each and every trial
- trials are independent of one another
- probability of having a success is a finite number, then

We say that the problem is based on the binomial distribution. In a problem like this, we define **X** as the random variable equals the number of successes obtained in n trials. Then **X t**akes the values 0, 1, 2, 3 . . . up to **n**. Therefore, one can view **X** a**s** a discrete random variable. Since number of ways of obtaining **k** successes in **n** trials may be achieved in $\binom{n}{k} = \frac{n!}{k!\ n-k\ !}$, therefore, binomial probability function may be formulated as $b(n, p, k) = \binom{n}{k} p^k q^{n-k}$.

Illustrative examples:

1. It is known that among the 10 telephone lines available in an office, the chance that any telephone is busy at an instant of time is 0.2. Find the probability that (i) exactly 3 lines are busy, (ii) What is the most probable number of busy lines and compute its probability, and (iii) What is the probability that all the telephones are busy?

Solution:

Here, the experiment about finding the number of busy telephone lines at an instant of time. Let X denotes the number of telephones which are active at a point of time, as there are **n = 10** telephones available; clearly X takes the values right from 0 up to 10. Let **p** denotes the chance of a telephone being busy, then it is given that **p = 0.2**, a finite value. The chance that a telephone line is free is **q = 0.8**. Since a telephone line being free or working is independent of one another, and since this value being same for each and every telephone line, we consider that this problem is based on binomial distribution. Therefore, the required probability mass function is

- $b(10, 0.2, k) = \binom{10}{k} \cdot (0.2)^k \cdot (0.8)^{(10-k)}$ Where k = 0, 1, 2 . . . 10.

(i) To find the chance that 3 lines are busy i.e. P[X = 3] = $b(10, 0.2, 3) = \binom{10}{3} \cdot (0.2)^3 \cdot (0.8)^7$

(ii) With p = 0.2, most probable number of busy lines is $n \cdot p = 10 \cdot 0.2 = 2$. The probability of this number equals $b(10, 0.2, 2) = \binom{10}{2} \cdot (0.2)^2 \cdot (0.8)^8$.

(iii) The chance that all the telephone lines are busy $= (0.2)^{10}$.

2. The chance that a bomb dropped from an airplane will strike a target is 0.4. 6 bombs are dropped from the airplane. Find the probability that (i) exactly 2 bombs strike the target? (ii) At least 1 strikes the target. (iii) None of the bombs hits the target?

Solution: Here, the experiment about finding the number of bombs hitting a target. Let X denotes the number of bombs hitting a target. As **n = 6** bombs are dropped from an airplane, clearly X takes the values right from 0 up to 6.

Let **p** denotes the chance that a bomb hits a target, then it is given that **p = 0.4**, a finite value. The chance that a telephone line is free is **q = 0.6**. Since a bomb dropped from airplane hitting a target or not is an independent event, and the probability of striking a target is same for all the bombs dropped from the plane, therefore one may consider that hat this problem is based on binomial distribution. Therefore, the required probability mass function is $b(10, 0.4, k) = \binom{10}{k} \cdot (0.4)^k \cdot (0.8)^{6-k}$.

(i) To find the chance that exactly 2 bombs hits a target,

i.e. **P[X = 2]** $= b(10, 0.4, 2) = \binom{10}{2} \cdot (0.4)^2 \cdot (0.8)^4$

(ii) Next to find the chance of the event, namely, at least 1 bomb hitting the target; i.e. $P[X \geq 1] = 1 - P[X < 1] = 1 - P[X = 0] = 1 - (0.6)^6$.

(iii) The chance that none of the bombs are going to hit the target is P[X=0] $= (0.6)^6$.

A discussion on Mean and Variance of Binomial Distribution Function

Let X be a discrete random variable following a binomial distribution function with the probability mass function given by $b(n, p, k) = \binom{n}{k} p^k q^{n-k}$. Consider the expectation of X, namely,

$$E(X) = \sum_{k=0}^{k=n} k \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k \cdot \frac{n!}{k!\ n-k\ !} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} \frac{n(n-1)!}{(k-1)!\ \ n-1+1-k\ \ !} pp^{k-1} q^{(n-1+1-k)}$$

$$= np \cdot \sum_{k=0}^{k=n} \frac{(n-1)!}{(k-1)![(n-1)! - (k-1)!]} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \cdot \sum_{k=1}^{k=n} \frac{(n-1)!}{(k-1)![(n-1)! - (k-1)!]} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \sum_{k=2}^{k=n} \binom{n-1}{k-1} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \cdot (p+q)^{n-1}$$

$$= np \text{ as } p+q=1$$

Thus, expected value of binomial distribution function is np.

To find variance of X, consider

$$E\ \ X^2\ \ = \sum_{k=0}^{k=n} k^2 \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k(k-1+1) \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k(k-1) \frac{n!}{k!\ \ n-k\ \ !} p^k q^{n-k} + \sum_{k=0}^{k=n} k \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} \frac{n(n-1)(n-2)!}{(k-2)![(n-2)! - (k-2)!]} p^2 p^{k-2} q^{[(n-2-(k-2)]} + E(X)$$

$$= n(n-1)p^2 \sum_{k=0}^{k=n} \frac{(n-2)!}{(k-2)![(n-2)! - (k-2)!]} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$$= n(n-1)p^2 \sum_{k=2}^{k=n} \frac{(n-2)!}{(k-2)![(n-2)! - (k-2)!]} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$$= n(n-1)p^2 \sum_{k=2}^{k=n} \binom{n-2}{k-2} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$= n(n-1)p^2(p+q)^{n-2} + np$. Since **p +q = 1**, it follows that

$$= n(n-1)p^2 + np$$

Therefore, $\mathrm{Var}(X) = E\left[X^2\right] - \left[E(X)\right]^2$

$$= n(n-1)p^2 + np - \left[np\right]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$= np - np^2 = np(1-p) = npq$. Hence, standard deviation of binomially distributed random variable is $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{npq}$.

A DISCUSSION ON POISSON DISTRIBUTION FUNCTION

This is a limiting case of the binomial distribution function. It is obtained by considering that the number of trials conducted is large and the probability of achieving a success in a single trial is very small i.e. here n is large and p is a small value. Therefore, Poisson distribution may be derived on the assumption that $n \to \infty$ and $p \to 0$. It is found that **Poisson distribution function** is

$$p(\lambda, k) = \frac{e^{-\lambda}\lambda^k}{k!}$$. Here, $\lambda = np$ and $k = 0, 1, 2, 3, \ldots. \infty$.

Expectation and Variance of a Poisson distribution function

Consider $E(X) = \sum_{k=0}^{k=\infty} k \cdot p(\lambda, k) = \sum_{k=0}^{k=\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$

$$= \sum_{k=0}^{k=\infty} \frac{e^{-\lambda}\lambda\lambda^{k-1}}{(k-1)!}$$

$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{k=\infty} \frac{\lambda^{k-1}}{(k-1)!}$. But $\sum_{k=1}^{k=\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{\lambda}$, therefore it follows that for a Poisson distribution function, $E(X) = \lambda$. Next to find Variance of X, first consider

$$E\left[X^2\right] = \sum_{k=0}^{k=\infty} k^2 \cdot p(\lambda, k)$$

$$= \sum_{k=0}^{k=\infty} k^2 \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1+1)\cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=0}^{k=\infty} k\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} + E(X)$$

$$= \sum_{k=0}^{k=\infty} \lambda^2 e^{-\lambda}\frac{\lambda^{k-2}}{(k-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda}\sum_{k=2}^{k=\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda$$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$. Thus, $E\ \ X^2\ \ = \lambda^2 + \lambda$. Hence, Variance of the Poisson distribution function is $\mathrm{Var}(X) = E\ \ X^2\ \ - \ E(X)\ \ ^2 = \lambda$. The standard deviation is $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\lambda}$

Illustrative Examples:

1. **It is known that the chance of an error in the transmission of a message through a communication channel is 0.002. 1000 messages are sent through the channel; find the probability that at least 3 messages will be received incorrectly.**

Solution: Here, the random experiment consists of finding an error in the transmission of a message. It is given that **n = 1000** messages are sent, a very large number, if **p** denote the probability of error in the transmission, we have **p = 0.002**, relatively a small number, therefore, this problem may be viewed as Poisson oriented. Thus, average number of messages with an error is $\lambda = np = 2$. Therefore, required probability function is $= p(2,k) = \frac{e^{-2}2^k}{k!}$, $k = 0, 1, 2, 3, \ldots \infty$. Here, the problem is about finding the probability of the event, namely,

$$P(X \geq 3) = 1 - P(X < 3) = 1 - \{P[X = 0] + P[X = 1] + P[X = 2]\}$$

$$= 1 - \left[\sum_{k=0}^{k=2} \frac{e^{-2}2^k}{k!}\right]$$

$$=1-e^{-2}\ 1+2+2\ =1-5e^{-2}$$

2. **A car hire –firm has two cars which it hires out on a day to day basis. The number of demands for a car is known to be Poisson distributed with mean 1.5. Find the proportion of days on which (i) There is no demand for the car and (ii) The demand is rejected.**

Solution: Here, let us consider that random variable X as the number of persons or demands for a car to be hired. Then X assumes the values 0, 1, 2, 3. … . . . It is given that problem follows a Poisson distribution with mean, $\lambda = 1.5$. Thus, required probability mass function may be written as $p(1.5, k) = \dfrac{e^{-1.5}(1.5)^k}{k!}$.

(i) Solution to I problem consists of finding the probability of the event, namely

P[X = 0] = $e^{-1.5}$**.**

(ii) The demand for a car will have to be rejected, when 3 or more persons approaches the firm seeking a car on hire. Thus, to find the probability of the event $P[X \geq 3]$.

Hence, $P[X \geq 3] = 1 - P\{X < 3\} = 1 - P[X = 0,1,2]\ = e^{-1.5}\left(1 + 1.5 + \dfrac{(1.5)^2}{2}\right)$.

<u>Illustrative examples based on Continuous Random Variable and it's Probability Density Function</u>

1. **Suppose that the error in the reaction temperature, in ^{O}C, for a controlled laboratory experiment is a R.V. X having the p.d.f**

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find (i) F(x) and (ii) use it to evaluate P (0<X≤1).

Solution: Consider $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

Case (i) x ≤ -1 $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0$

Case (ii) -1< x < 2

$$F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{-1} f(t)dt+\int_{-1}^{x} f(t)dt=0+\int_{-1}^{x}\frac{t^2}{3}dt=\left.\frac{t^3}{9}\right|_{-1}^{x}=\frac{x^3+1}{9}.$$

Case (iii) x = 2 $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{-1} f(t)dt+\int_{-1}^{2} f(t)dt+\int_{2}^{x} f(t)dt$

$$=0+\int_{-1}^{2}\frac{t^2}{3}dt+0=\left.\frac{t^3}{9}\right|_{-1}^{2}=\frac{8+1}{9}=1.$$ **Therefore,**

$$F(x)=\begin{cases}0, & x\le -1\\ \dfrac{x^3+1}{9}, & -1<x<2.\\ 1, & x\ge 2.\end{cases}$$

2. If the p.d.f of a R.V. X having is given by $f(x)=\begin{cases}2kxe^{-x^2}, & for\ x>0\\ 0, & for\ x\le 0.\end{cases}$

Find (a) the value of *k* and (b) distribution function F(X) for X.

WKT $\int_{0}^{\infty} 2kxe^{-x^2}dx=1$

$\Rightarrow \int_{0}^{\infty} ke^{-t}dt=1(\text{put } x^2=t)$

$\Rightarrow \left. ke^{-t}\right|_{0}^{\infty}=1$

$\Rightarrow (0+k)=1 \Rightarrow k=1$

$$F(x)=P(X\le x)=\int_{-\infty}^{x} f(t)dt=0,\quad \text{if } x\le 0$$

$$=\int_{-\infty}^{0} f(t)dt+\int_{0}^{x} f(t)dt,\quad \text{if } x>0$$

$$=0+\int_{0}^{x} 2te^{-t^2}dt=(-e^{-z})_{0}^{x^2}=(1-e^{-x^2}).$$

www.rejinpaul.com

$$F(x) = \begin{cases} 1 - e^{-x^2}, & \text{for } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

3. Find the C.D.F of the R.V. whose P.D.F is given by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{for } 0<x \leq 1 \\ \frac{1}{2}, & \text{for } 1<x \leq 2 \\ \frac{3-x}{2}, & \text{for } 2<x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Solution: Case (i) x ≤ 0 $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt = 0$

Case (ii) 0 < x ≤ 1 $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{x} \frac{t}{2} dt = \frac{x^2}{4}$

Case (iii) 1 < x ≤ 2 $F(x) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt$

$$= 0 + \int_{0}^{1} \frac{t}{2} dt + \int_{1}^{x} \frac{1}{2} dt = \frac{2x-1}{4}$$

Case (iv) 2 < x ≤ 3 $F(x) = \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{2} f(t)dt + \int_{2}^{x} f(t)dt$

$$F(x) = \frac{6x - x^2 - 5}{4}$$

Case (v) for x >3, F(x) = 1. Therefore,

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{x^2}{4}, & \text{if } 0<x \leq 1 \\ \frac{2x-1}{4}, & \text{if } 1<x \leq 2 \\ \frac{6x-x^2-5}{4}, & \text{if } 2<x \leq 3 \\ 1, & \text{if } x>3 \end{cases}$$

4. The trouble shooting of an I.C. is a R.V. X whose distribution function is given by $F(x)=\begin{cases} 0, & for\ x\le 3 \\ 1-\dfrac{9}{x^2}, & for\ x>3. \end{cases}$

If *X* denotes the number of years, find the probability that the I.C. will work properly

(a) less than 8 years
(b) beyond 8 years
(c) anywhere from 5 to 7 years
(d) Anywhere from 2 to 5 years.

Solution: We have $F(x)=\int_0^x f(t)dt=\begin{cases} 0, & for\ x\le 3 \\ 1-\dfrac{9}{x^2}, & for\ x>3. \end{cases}$

For (a): $P(x\le 8)=\int_0^8 f(t)dt=1-\dfrac{9}{8^2}=0.8594$

For Case (b): P(x > 8) = 1 – P(x≤ 8) = 0.1406

For Case (c): P (5≤ x≤ 7) = F (7) – F (5) = (1-9/7^2) – (1-9/5^2) = 0.1763

For Case (d): P (2≤ x≤ 5) = F (5) – F (2) = (1-9/5^2) – (0) = 0.64

5.* A continuous R.V. X has the distribution function is given *by

$$F(x)=\begin{cases} 0, & x\le 1 \\ c(x-1)^4, & 1\le x\le 3 \\ 1, & x>3. \end{cases}$$

Find c and the probability density function.

Solution: We know that $f(x)=\dfrac{d}{dx}[F(x)]$

$$\therefore\quad f(x)=\begin{cases} 0, & x\le 1 \\ 4c(x-1)^3, & 1\le x\le 3 \\ 0, & x>3. \end{cases}$$

$$\therefore\quad f(x)=\begin{cases} 4c(x-1)^3, & 1\le x\le 3 \\ 0, & \text{otherwise} \end{cases}$$

Since we must have $\int_{-\infty}^{\infty} f(x)dx = 1$,

$$\int_1^3 4c(x-1)^3 dx = 1 \quad \Rightarrow \left[c(x-1)^4\right]_1^3 = 1$$

$$\Rightarrow 16c = 1 \quad \therefore c = \frac{1}{16}$$

Using this, one can give the probability function just by substituting the value of c above.

A discussion on some standard distribution functions of continuously distributed random variable:

This distribution, sometimes called the negative exponential distribution, occurs in applications such as reliability theory and queuing theory. Reasons for its use include its memory less (Markov) property (and resulting analytical tractability) and its relation to the (discrete) Poisson distribution. Thus, the following random variables may be modeled as exponential:

- Time between two successive job arrivals to a computing center (often called inter-arrival time)
- Service time at a server in a queuing network; the server could be a resource such as CPU, I/O device, or a communication channel
- Time to failure of a component i.e. life time of a component
- Time required repairing a component that has malfunctioned.

The exponential distribution function is given by, $f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

The probability distribution function may be written as $F(x) = \int_{-\infty}^{x} f(x)dx$ which may be computed as $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$.

www.rejinpaul.com

Mean and Variance of Exponential distribution function

Consider mean $(\mu) = \int_{-\infty}^{\infty} x\cdot f(x)dx = \int_{0}^{\infty} x\cdot\lambda e^{-\lambda x}dx$

$$= \lambda\left[x\cdot\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\cdot\left(\frac{e^{-\lambda x}}{-\lambda^2}\right)\right]_0^{\infty} = -\lambda\left[0 - \frac{1}{\lambda^2}\right] = \frac{1}{\lambda}$$

Consider E $X^2 = \int_{-\infty}^{\infty} x^2\cdot f(x)dx = \int_{0}^{\infty} x^2\cdot\lambda e^{-\lambda x}dx$

$$= \lambda\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{-\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty} = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E\ X^2 - E(X)^2 = \frac{1}{\lambda^2}.$$

The standard deviation is $\sigma = \sqrt{Var(X)} = \frac{1}{\lambda}$.

<u>Illustrative examples based on Exponential distribution function</u>

1. The duration of telephone conservation has been found to have an exponential distribution with mean 2 minutes. Find the probabilities that the conservation may last (i) more than 3 minutes, (ii) less than 4 minutes and (iii) between 3 and 5 minutes.

Solution: Let X denotes the random variable equals number of minute's conversation may last. It is given that X is exponentially distributed with mean 3 minutes. Since for an exponential distribution function, mean is known to be $\frac{1}{\lambda}$, so $\frac{1}{\lambda} = 2$ or $\lambda=0.5$. The Probability density function can now be written as $f(x) = \begin{cases} 0.5e^{-0.5x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$.

(i) To find the probability of the event, namely,

$$P[X > 3] = 1 - P[X \le 3] = 1 - \int_0^3 0.5e^{-0.5x}dx$$

(ii) To find the probability of the event, namely $P[X < 4] = \int_0^4 0.5e^{-0.5x}dx$.

(iii) To find the probability of the event $P[3 < X < 5] = \int_3^5 0.5e^{-0.5x}dx$.

2. in a town, the duration of a rain is exponentially distributed with mean equal to 5 minutes. What is the probability that (i) the rain will last not more than 10 minutes (ii) between 4 and 7 minutes and (iii) between 5 and 8 minutes?

Solution: An identical problem to the previous one. Thus, may be solved on similar lines.

Discussion on Gaussian or Normal Distribution Function

Among all the distribution of a continuous random variable, the most popular and widely used one is normal distribution function. Most of the work in correlation and regression analysis, testing of hypothesis, has been done based on the assumption that problem follows a normal distribution function or just everything normal. Also, this distribution is extremely important in statistical applications because of the central limit theorem, which states that under very general assumptions, the mean of a sample of n mutually Independent random variables (having finite mean and variance) are normally distributed in the limit $n \to \infty$. It has been observed that errors of measurement often possess this distribution. Experience also shows that during the wear – out phase, component life time follows a normal distribution. The purpose of today's lecture is to have a detailed discussion on the same.

The normal density function has well known bell shaped curve which will be shown on the board and it may be given as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$ where $-\infty < \mu < \infty$ and $\sigma > 0$. It will be shown that μ and σ are respectively denotes mean and variance of the normal distribution. As the probability or cumulative distribution function, namely, $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$ has no closed form, evaluation of integral in an interval is difficult. Therefore, results relating to probabilities are computed numerically and recorded in special table called normal distribution table. However, It pertain to the standard normal distribution function by choosing μ and σ and their entries are values of the function, $F_z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$. Since the standard normal distribution is symmetric, it can be shown that $F_z(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} f(t)\, dt = 1 - F_Z(z)$.

Thus, tabulations are done for positive values of z only. From this it is clear that

- $P(a \leq X \leq b) = F(b) - F(a)$
- $P(a < X < b) = F(b) - F(a)$
- $P(a < X) = 1 - P(X \leq a) = 1 - F(a)$

Note: Let X be a normally distributed random variable taking a particular value, x, the corresponding value of the standardized variable is given by $z = \frac{x - \mu}{\sigma}$. Hence,

$$F(x) = P(X \leq x) = F_Z\left(\frac{x-\mu}{\sigma}\right).$$

Illustrative Examples based on Normal Distribution function:

1. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2150 hours, (b) less than 1940 hours and (c) more than 1920 hours and but less than 2060 hours.

Solution:

Here, the experiment consists of finding the life of electric bulbs of a particular make (measured in hours) from a lot of 2000 bulbs. Let X denotes the random variable equals the life of an electric bulb measured in hours. It is given that X follows normal distribution with mean $\mu = 2040$ hours and $\sigma = 60$ hours.

First to calculate $\boldsymbol{P}(\boldsymbol{X} > 2150 \text{ hours}) = 1 - \boldsymbol{P}(\boldsymbol{X} \le 2150)$

$$= 1 - \boldsymbol{F}_{\boldsymbol{z}}\ 1.8333 = 1 - 0.9664 = 0.0336$$

Therefore, number of electrical bulbs with life expectancy more than 2150 hours is $0.0336 \times 2000 \approx 67$.

Next to compute the probability of the event; $\boldsymbol{P}(\boldsymbol{X} < 1950 \text{ hours}) = \mathrm{F}_{\mathrm{Z}}\left(\frac{1950-2040}{60}\right)$

$$= \mathrm{F}_{\mathrm{Z}}\ -1.5 = 1 - \boldsymbol{F}_{\boldsymbol{z}}(1.5) = 1 - 0.9332 = 0.0668$$

Therefore, in a lot of 2000 bulbs, number of bulbs with life expectancy less than 1950 hours is 0.0668 * 2000 = 134 bulbs.

Finally, to find the probability of the event, namely,

$$\boldsymbol{P}(1920 < \boldsymbol{X} < 2060) = \boldsymbol{F}(2060) - \boldsymbol{F}(1920)$$

$$= \boldsymbol{F}_{\boldsymbol{z}}\left(\frac{2060-2040}{60}\right) - \boldsymbol{F}_{\boldsymbol{z}}\left(\frac{1920-2040}{60}\right)$$

$$= \boldsymbol{F}_{\boldsymbol{z}}\ 0.3333 - \boldsymbol{F}_{\boldsymbol{z}}\ -2$$

$$= \boldsymbol{F}_{\boldsymbol{z}}\ 0.3333 - 1 + \boldsymbol{F}_{\boldsymbol{z}}\ 2$$

$$= 0.6293 - 1 + 0.9774 = 0.6065.$$

Therefore, number of bulbs having life anywhere in between 1920 hours and 2060 hours is 0.6065 * 2000 = 1213.

2. Assume that the reduction of a person's oxygen consumption during a period of Transcendenta Meditation (T.M.) is a continuous random variable X normally distributed with mean 37.6 cc/min and S.D. 4.6 cc/min. Determine the probability that during a period of T.M. a person's oxygen consumption will be reduced by (a) at least 44.5 cc/min (b) at most 35.0 cc/min and (c) anywhere from 30.0 to 40.0 cc/min.

Solution: Here, X a random variable is given to be following normal distribution function with mean.$P \ \mu = 37.6$ and $\sigma = 4.6$. Let us consider that X as the random equals the rejection of oxygen consumption during T M period and measured in cc/min.

(i) To find the probability of the event $P[X \geq 44.5] = 1 - F(44.5)$

$$= 1 - F_z\left(\frac{44.5 - 37.6}{4.6}\right)$$

$$= 1 - F_z \ 1.5$$

$$= 1 - 0.9332 = 0.0668.$$

(ii) To find the probability of the event, $P[X \leq 35.0] = F(33.5)$

$$= F_z\left(\frac{35.0 - 37.6}{4.6}\right)$$

$$= F_z \ -0.5652$$

$$= 1 - F_z \ 0.5652$$

$$= 1 - 0.7123 = 0.2877.$$

(iii) Consider the probability of the event $P[30.0 < X < 40.0]$

$$= F(40) - F(30)$$

$$= F_z\left(\frac{40 - 37.6}{4.6}\right) - F_z\left(\frac{30 - 37.6}{4.6}\right)$$

$$= F_z(0.5217) - F_z(-1.6522)$$

$$= 0.6985 - 1 + 0.9505 = 0.6490$$

3. An analog signal received at a detector (measured in micro volts) may be modeled as a Gaussian random variable N (200, 256) at a fixed point in time. What is the probability that the signal will exceed 240 micro volts? What is the probability that the signal is larger than 240 micro volts, given that it is larger than 210 micro volts?

Solution: Let X be a CRV denotes the signal as detected by a detector in terms of micro volts. Given that X is normally distributed with mean 200 micro volts and variance 256 micro volts. To find the probability of the events, namely, (i) P (X > 240 micro volts] and (ii) P[X > 240 micro volts | X > 210 micro volts].

Consider $P[X > 240] = 1 - P[X \le 240]$

$$= 1 - F(240)$$

$$= 1 - F_z\left(\frac{240-200}{16}\right)$$

$$= 1 - F_z \quad 2.5$$

= 1 – 0.9938

= 0.00621

Next consider P**[X > 240 | X > 210]**

$$= \frac{P[X > 240 \text{ and X} > 210]}{P[X > 210]}$$

$$= \frac{P[X > 240\ ]}{P[X > 210]} = \frac{1 - P[X \le 240\ ]}{1 - P[X \le 210]}$$

$$= \frac{1 - F_z\left[\frac{240-200}{16}\right]}{1 - F_z\left[\frac{210-200}{16}\right]} = \frac{1 - F_z(2.5)}{1 - F_z(0.625)}$$

$$= \frac{1 - 0.9939}{1 - 0.73401} = 0.2335$$

www.rejinpaul.com

Unit VIII
Sampling Theory

Statistical Inference is a branch of Statistics which uses probability concepts to deal with uncertainty in decision making. There are a number of situations where in we come across problems involving decision making. For example, consider the problem of buying 1 kilogram of rice, when we visit the shop, we do not check each and every rice grains stored in a gunny bag; rather we put our hand inside the bag and collect a sample of rice grains. Then analysis takes place. Based on this, we decide to buy or not. Thus, the problem involves studying whole rice stored in a bag using only a sample of rice grains.

This topic considers two different classes of problems

1. Hypothesis testing – we test a statement about the population parameter from which the sample is drawn.
2. Estimation – A statistic obtained from the sample collected is used to estimate the population parameter.

First what is meant by hypothesis testing?

This means that testing of hypothetical statement about a parameter of population.

Conventional approach to testing:

The procedure involves the following:

1. First we set up a definite statement about the population parameter which we call it as null hypothesis, denoted by H_0. According to Professor R. A. Fisher,

Null Hypothesis is the statement which is tested for possible rejection under the assumption that it is true. Next we set up another hypothesis called alternate statement which is just opposite of null statement; denoted by H_1 which is just complimentary to the null hypothesis. Therefore, if we start with $H_0 : \mu = \mu_0$ then alternate hypothesis may be considered as either one of the following statements; $H_1 : \mu \neq \mu_0$, or $H_1 : \mu > \mu_0$ or $H_1 : \mu < \mu_0$.

As we are studying population parameter based on some sample study, one can not do the job with 100% accuracy since sample is drawn from the population and possible sample may not represent the whole population. Therefore, usually we conduct analysis at certain level of significance (lower than 100%. The possible choices include 99%, or 95% or 98% or 90%. **Usually we conduct analysis at 99% or 95% level of significance,** denoted by the symbol α . We test H_0 against H_1 at certain level of significance. The confidence with which a person rejects or accepts H_0 depends upon the significance level adopted. It is usually expressed in percentage forms such as 5% or 1% etc. Note that when α is set as 5%, then probability of rejecting null hypothesis when it is true is only 5%. It also means that when the hypothesis in question is accepted at 5% level of significance, then statistician runs the risk of taking wrong decisions, in the long run, is only 5%. The above is called II step of hypothesis testing.

Critical values or Fiducial limit values for a two tailed test:

Sl. No	Level of significance	Theoretical Value
1	$\alpha = 1\%$	2.58
2	$\alpha = 2\%$	2.33
3	$\alpha = 5\%$	1.96

<u>Critical values or Fiducial limit values for a single tailed test (right and test)</u>

Tabulated value	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
Right – tailed test	2.33	1.645	1.28
Left tailed test	-2.33	-1.645	-1.28

<u>Setting a test criterion</u>: The third step in hypothesis testing procedure is to construct a test criterion. This involves selecting an appropriate probability distribution for the particular test i.e. a proper probability distribution function to be chosen. Some of the distribution functions used are t, F, when the sample size is small (size lower than 30).

However, for large samples, normal distribution function is preferred. Next step is the computation of statistic using the sample items drawn from the population. Usually, samples are drawn from the population by a procedure called random, where in each and every data of the population has the same chance of being included in the sample. Then the computed value of the test criterion is compared with the tabular value; as long the calculated value is lower then or equal to tabulated value, we accept the null hypothesis, otherwise, we reject null hypothesis and accept the alternate hypothesis. Decisions are valid only at the particular level significance of level adopted.

During the course of analysis, there are two types of errors bound to occur. These are **(i) Type – I error and (ii) Type – II error.**

Type – I error: This error usually occurs in a situation, when the null hypothesis is true, but we reject it i.e. rejection of a correct/true hypothesis constitute type I error.

Type – II error: Here, null hypothesis is actually false, but we accept it. Equivalently, accepting a hypothesis which is wrong results in a type – II error. The probability of committing a type – I error is denoted by α where

α = Probability of making type I error = Probability [Rejecting H_0 | H_0 is true]

On the other hand, type – II error is committed by not rejecting a hypothesis when it is false. The probability of committing this error is denoted by β . Note that

β = Probability of making type II error = Probability [Accepting H_1 | H_1 is false]

Critical region:

A region in a sample space S which amounts to Rejection of H_0 is termed as critical region.

One tailed test and two tailed test:

This depends upon the setting up of both null and alternative hypothesis.

A note on computed test criterion value:

1. When the sampling distribution is based on population of proportions/Means, then test criterion may be given as

$$Z_{cal} = \frac{\text{Expected results - Observed results}}{\text{Standard error of the distribution}}$$

Application of standard error:

1. S.E. enables us to determine the probable limit within which the population parameter may be expected to lie. For example, the probable limits for population of proportion are given by $\boldsymbol{p} \pm 3\sqrt{\boldsymbol{pqn}}$. Here, p represents the chance of achieving a success in a single trial, q stands for the chance that there is a failure in the trial and n refers to the size of the sample.
2. The magnitude of standard error gives an index of the precision of the parameter.

ILLUSTRATIVE EXAMPLES

1. A coin is tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is un– biased?

Solution: First we construct null and alternate hypotheses set up $\boldsymbol{H}_0$: The coin is not a biased one. Set up $\boldsymbol{H}_1$: Yes, the coin is biased. As the coin is assumed be fair and it is tossed 400 times, clearly we must expect 200 times heads occurring and 200 times tails. Thus, expected number of heads is 200. But the observed result is 216. There is a difference of 16. Further, standard error is $\boldsymbol{\sigma} = \sqrt{\boldsymbol{npq}}$. With p = ½, q = ½ and n = 400, clearly $\boldsymbol{\sigma} = 10$. The test criterion is $\boldsymbol{z}_{\text{cal}} = \dfrac{\text{difference}}{\text{standard error}} = \left|\dfrac{216-240}{10}\right| = 1.6$

If we choose $\boldsymbol{\alpha} = 5\%$, then the tabulated value for a two tailed test is 1.96. Since, the calculated value is lower than the tabulated value; we accept the null hypothesis that coin is un – biased.

2. **A person throws a 10 dice 500 times and obtains 2560 times 4, 5, or 6. Can this be attributed to fluctuations in sampling?**

Solution: As in the previous problem first we shall set up $\boldsymbol{H}_0$: The die is fair and $\boldsymbol{H}_1$: The die is unfair. We consider that problem is based on a two – tailed test. Let us choose level of significance as $\boldsymbol{\alpha} = 5\%$ then, the tabulated value is 1.96. Consider computing test criterion, $\boldsymbol{z}_{\text{cal}} = \left|\dfrac{\text{Expected value - observed result}}{\text{standard error}}\right|$; here, as the dice is tossed by a person 5000 times, and on the basis that die is fair, then chance of getting any of the 6 numbers is 1/6. Thus, chance of getting either 4 or 5, or 6 is p = ½. Also, q = ½. With n = 5000, standard error, $\sigma = \sqrt{npq}$ = 35.36. Further, expected value of obtaining 4 or 5 or 6 is 2500. Hence, $z_{\text{cal}} = \left|\frac{2500 - 2560}{35.36}\right| = 1.7$ which is lower than 1.96. Hence, we conclude that die is a fair one.

3. A sample of 1000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to the percentage of foggy days in the district?

Solution: Let **p** denote the probability that a day is foggy in nature in a district as reported by meteorological records. Clearly, $p = \frac{120}{1000} = 0.12$ and **q** = 0.88. With **n** = 1000, the probable limits to the percentage of foggy days is given by $p \pm 3\sqrt{pqn}$. Using the data available in this problem, one obtains the answer as $0.12 \pm 3\sqrt{0.12 \cdot 88 \cdot 1000}$. Equivalently, 8.91% to 15.07%.

4. A die was thrown 9000 times and a throw of 5 or 6 was obtained 3240 times. On the assumption of random throwing, do the data indicate that die is biased? (Model Question Paper Problem)

Solution: We set up the null hypothesis as $H_0: \text{Die is un-biased.}$ Also, $H_1: \text{Die is biased.}$. Let us take level of significance as $\alpha = 5\%$. Based on the assumption that distribution is normally distributed, the tabulated value is 1.96. The chance of getting each of the 6 numbers is same and it equals to 1/6 therefore chance of getting either 5 or 6 is 1/3. In a throw of 9000 times, getting the numbers either 5 or 6 is $\frac{1}{3} \times 9000 = 3000$. Now the difference in these two results is 240. With **p** = 1/3, **q** = 2/3, **n** = 9000, $S.E. = \sqrt{npq}$ = 44.72. Now consider the test criterion $z_{\text{cal}} = \frac{\text{Difference}}{\text{S.E.}}$ = $= \frac{240}{44.72} = 5.367$ which is again more than the tabulated value. Therefore, we reject null hypothesis and accept the alternate that die is highly biased.

<u>Tests of significance for large samples:</u>

In the previous section, we discussed problems pertaining to sampling of attributes. It is time to think of sampling of other variables one may come across in a practical situation such as height weight etc. We say that a sample is small when the size is usually lower than 30, otherwise it is called a large one.

The study here is based on the following assumptions: (i) the random sampling distribution of a statistic is approximately normal and (ii) values given by the samples are sufficiently close to the population value and can be used in its place for calculating standard error. When the standard deviation of population is known, then $\text{S.E}(\overline{\text{X}}) = \dfrac{\sigma_p}{\sqrt{n}}$ where σ_p denotes the standard deviation of population. When the standard deviation of the population is unknown, then $\text{S.E}(\overline{\text{X}}) = \dfrac{\sigma}{\sqrt{n}}$ where σ is the standard deviation of the sample.

Fiducial limits of population mean are:

95% fiducial limits of population mean are $\overline{\text{X}} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$

99% fiducial limits of population mean are $\overline{X} \pm 2.58\dfrac{\sigma}{\sqrt{n}}$. Further, test criterion $z_{\text{cal}} = \left|\dfrac{\overline{x} - \mu}{\text{S.E.}}\right|$

ILLUSTRATIVE EXAMPLES

1. **A sample of 100 tyres is taken from a lot. The mean life of tyres is found to be 39, 350 kilo meters with a standard deviation of 3, 260. Could the sample come from a population with mean life of 40, 000 kilometers? Establish 99% confidence limits within which the mean life of tyres is expected to lie.**

Solution: First we shall set up null hypothesis, $H_0 : \mu = 40,000$, alternate hypothesis as $H_1 : \mu \neq 40,000$. We consider that the problem follows a two tailed test and chose $\alpha = 5\%$. Then corresponding to this, tabulated value is 1.96. Consider the expression for finding test criterion, $z_{\text{cal}} = \left|\dfrac{\overline{x} - \mu}{\text{S.E.}}\right|$. Here, μ=40, 000, $\overline{x} = 39, 350$ and

$\sigma = 3,260$, **n = 100.** $\text{S.E.} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{3,260}{\sqrt{100}} = 326$. Thus, $z_{cal} = 1.994$. As this value is slightly greater than 1.96, we reject the null hypothesis and conclude that sample has not come from a population of 40, 000 kilometers.

The 99% confidence limits within which population mean is expected to lie is given as $\bar{x} \pm 2.58 \times \text{S.E.}$ i.e. $39,350 \pm 2.58 \times 326 = (38,509, 40,191)$.

2. **The mean life time of a sample of 400 fluorescent light bulbs produced by a company is found to be 1, 570 hours with a standard deviation of 150 hours. Test the hypothesis that the mean life time of bulbs is 1600 hours against the alternative hypothesis that it is greater than 1, 600 hours at 1% and 5% level of significance.**

Solution: First we shall set up null hypothesis, $H_0 : \mu = 1,600$ hours, alternate hypothesis as $H_1 : \mu > 1,600$ hours. We consider that the problem follows a two tailed test and chose $\alpha = 5\%$. Then corresponding to this, tabulated value is 1.96. Consider the expression for finding test criterion, $z_{cal} = \left| \dfrac{\bar{x} - \mu}{\text{S.E.}} \right|$. Here, $\mu = 1,600$, $\bar{x} = 1,570$, $n = 400$, $\sigma = 150$ hours so that using all these values above, it can be seen that $z_{cal} = 4.0$ which is really greater than 1.96. Hence, we have to reject null hypothesis and to accept the alternate hypothesis.

Test of significance of difference between the means of two samples

Consider two populations P1 and P2. Let S_1 and S_2 be two samples drawn at random from these two different populations. Suppose we have the following data about these two samples, say

Samples/Data	Sample size	Mean	Standard Deviation
S_1	n_1	$\bar{x}_1$	σ_1
S_2	n_2	$\bar{x}_2$	σ_2

then standard error of difference between the means of two samples S_1 and S_2 is $\text{S.E} = \sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}$ and the test criterion is $Z_{cal} = \dfrac{\text{Difference of sample means}}{\text{Standard error}}$. The rest of the analysis is same as in the preceding sections.

When the two samples are drawn from the same population, then standard error is

$\text{S.E} = \sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$ and test criterion is $Z_{cal} = \dfrac{\text{Difference of sample means}}{\text{Standard error}}$.

When the standard deviations are un – known, then standard deviations of the two samples must be replaced. Thus, $\text{S.E} = \sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}$ where s_1 and s_2 are standard deviations of the two samples considered in the problem.

ILLUSTRATIVE EXAMPLES

1. Intelligence test on two groups of boys and girls gave the following data:

Data	Mean	Standard deviation	Sample size
Boys	75	15	150
Girls	70	20	250

Is there a significant difference in the mean scores obtained by boys and girls?

Solution: We set up null hypothesis as H_0 : there is no significant difference between the mean scores obtained by boys and girls. The alternate hypothesis is considered as H_1 : Yes, there is a significant difference in the mean scores obtained by boys and girls. We choose level of significance as $\alpha = 5\%$ so that tabulated value is 1.96. Consider $z_{cal} = \dfrac{\text{Difference of means}}{\text{Standard Error}}$. The standard error may be calculated as $\text{S.E} = \sqrt{\dfrac{15^2}{150} + \dfrac{20^2}{250}} = 1.761$, The test criterion is $z_{cal} = \dfrac{75 - 70}{1.761} = 2.84$. As 2.84 is more than 1.96, we have to reject null hypothesis and to accept alternate hypothesis that there are some significant difference in the mean marks scored by boys and girls.

2. A man buys 50 electric bulbs of "Philips" and 50 bulbs of "Surya". He finds that Philips bulbs give an average life of 1,500 hours with a standard deviation of 60 hours and Surya bulbs gave an average life of 1, 512 hours with a standard deviation of 80 hours. Is there a significant difference in the mean life of the two makes of bulbs?

Solution: we set up null hypothesis, $\boldsymbol{H_0}$: there is no significant difference between the bulbs made by the two companies, the alternate hypothesis can be set as $\boldsymbol{H_1}$: Yes, and there could be some significant difference in the mean life of bulbs. Taking $\alpha = 1\%$ and $\alpha = 5\%$, the respective tabulated values are 2.58 and 1.96. Consider standard error is $\text{S.E} = \sqrt{\frac{60^2}{50} + \frac{80^2}{50}} = 14.14$ **so** that $z_{cal} = \frac{1512 - 1500}{14.14} = 0.849$. Since the calculated value is certainly lower than the two tabulated values, we accept the hypothesis there is no significant difference in the make of the two bulbs produced by the companies.

A discussion on tests of significance for small samples

So far the problem of testing a hypothesis about a population parameter was based on the assumption that sample drawn from population is large in size (more than 30) and the probability distribution is normally distributed. However, when the size of the sample is small, (say < 30) tests considered above are not suitable because the assumptions on which they are based generally do not hold good in the case of small samples. IN particular, here one cannot assume that the problem follows a normal distribution function and those values given by sample data are sufficiently close to the population values and can be used in their place for the calculation of standard error. Thus, it is a necessity to develop some alternative strategies to deal with problems having sample size relatively small. Also, we do see a number of problems involving small samples. With these in view, here, we will initiate a detailed discussion on the same.

Here, too, the problem is about testing a statement about population parameter; i.e. in ascertaining whether observed values could have arisen by sampling fluctuations from some value given in advance. For example, if a sample of 15 gives a correlation coefficient of +0.4, we shall be interested not so much in the value of the correlation in the parent population, but more generally this value could have come from an un – correlated population, i.e. whether it is significant in the parent population. It is widely accepted that when we work with small samples, estimates will vary from sample to sample.

Further, in the theory of small samples also, we begin study by making an assumption that parent population is normally distributed unless otherwise stated. Strictly, whatever the decision one takes in hypothesis testing problems is valid only for normal populations.

Sir William Gosset and R. A. Fisher have contributed a lot to theory of small samples. Sir W. Gosset published his findings in the year 1905 under the pen name “student”. He gave a test popularly known as “t – test” and Fisher gave another test known as “z – test”. These tests are based on “t distribution and “z – distribution”.

Student's t - distribution function

Gosset was employed by the Guinness and Son, Dublin bravery, Ireland which did not permit employees to publish research work under their own names. So Gosset adopted the pen name "student" and published his findings under this name. Thereafter, the t – distribution commonly called student's t – distribution or simply student's distribution.

The t – distribution to be used in a situation when the sample drawn from a population is of size lower than 30 and population standard deviation is un – known. The t – statistic, t_{cal} is defined as $\gamma = n - 1 = 12$ $t_{\text{cal}} = \left(\frac{\bar{x} - \mu}{S}\right) \cdot \sqrt{n}$ where $S = \sqrt{\frac{\sum_{i=1}^{i=n} \left(x_i - \bar{x}\right)^2}{n-1}}$, $\bar{x}$ is the sample mean, **n** is the sample size, and x_i are the data items.

The t – distribution function has been derived mathematically under the assumption of a normally distributed population; it has the following form

$f(t) = C\left(1 + \frac{t^2}{\gamma}\right)^{-\left(\frac{\gamma+1}{2}\right)}$ where C is a constant term and γ = **n - 1** denotes the number of degrees of freedom. As the p.d.f. of a t – distribution is not suitable for analytical treatment. Therefore, the function is evaluated numerically for various values of t, and for particular values of γ . The t – distribution table normally given in statistics text books gives, over a range of values of γ, the probability values of exceeding by chance value of t at different levels of significance. The t – distribution function has a different value for each degree of freedom and when degrees of freedom approach a large value, t – distribution is equivalent to normal distribution function.

The application of t – distribution includes (i) testing the significance of the mean of a random sample i.e. determining whether the mean of a sample drawn from drawn from a normal population deviates significantly from a stated value (i.e. hypothetical value of the populations mean) and (ii) testing whether difference between means of two independent samples is significant or not i.e. ascertaining whether the two samples comes from the same normal population? (iii) Testing difference between means of two dependent samples is significant? (iv) Testing the significance of on observed correlation coefficient.

Procedures to be followed in testing a hypothesis made about the population parameter using student's t - distribution:

- As usual first set up null hypothesis,
- Then, set up alternate hypothesis,
- Choose a suitable level of significance,

- Note down the sample size, n and the number of degrees of freedom,
- Compute the theoretical value, t_{tab} by using t – distribution table.
- t_{tab} value is to be obtained as follows: If we set up $\alpha = 5\% = 0.05$, suppose $\gamma = 9$ then, t_{tab} is to be obtained by looking in 9th row and **in the column** $\alpha = 0.025$ (i.e. half of α = 0.05) **.**
- The test criterion is then calculated using the formula, $t_{\text{cal}} = \left(\frac{\bar{x} - \mu}{s}\right) \cdot \sqrt{n}$
- Later, the calculated value above is compared with tabulated value. As long as the calculated value matches with the tabulated value, we as usual accept the null hypothesis and on the other hand, when the calculated value becomes more than tabulated value, we reject the null hypothesis and accept the alternate hypothesis.

ILLUSTRATIVE EXAMPLES

1. The manufacturer of a certain make of electric bulbs claims that his bulbs have a mean life of 25 months with a standard deviation of 5 months. Random samples of 6 such bulbs have the following values: Life of bulbs in months: 24, 20, 30, 20, 20, and 18. Can you regard the producer's claim to valid at 1% level of significance? (Given that $\text{t}_{\text{tab}} = 4.032$ **corresponding to** $\gamma = 5$**).**

Solution: To solve the problem, we first set up the null hypothesis $H_0 : \mu = 25 \text{ months}$, alternate hypothesis may be treated as $H_0 : \mu < 25 \text{ months}$. To set up $\alpha = 1\%$**,** then tabulated value corresponding to this level of significance is $t_{\text{tab}}\,|_{\alpha=1\% \text{ and } \gamma=5} = 4.032$ (4.032 value has been got by looking in the 5^{th} row) . The test criterion is given by $t_{\text{cal}} = \left(\frac{\bar{x} - \mu}{s}\right) \cdot \sqrt{n}$ where $s = \sqrt{\frac{\sum_{i=1}^{i=n} \left(x_i - \bar{x}\right)^2}{n-1}}$.

Consider

x_i	$\bar{x}$	$x_i - \bar{x}$	$\left(x_i - \bar{x}\right)^2$
24		1	1

www.rejinpaul.com

26		3	9
30		7	49
20	**23**	-3	9
20		-3	9
18		-5	25
Total = 138		-	**Total = 102**

Thus, $S = \sqrt{\dfrac{102}{5}} = \sqrt{20.4} = 4.517$ and $t_{\text{cal}} = \left|\dfrac{23-25}{4.517}\right|\sqrt{6} = 1.084$. Since the calculated value, 1.084 is lower than the tabulated value of 4.032; we accept the null hypothesis as mean life of bulbs could be about 25 hours.

2. A certain stimulus administered to each of the 13 patients resulted in the following increase of blood pressure: 5, 2, 8,-1, 3, 0, -2, 1, 5, 0, 4, 6, 8. Can it be concluded that the stimulus, in general, be accompanied by an increase in the blood pressure? (Model Question Paper Problem)

Solution: We shall set up $H_0 : \mu_{\text{before}} = \mu_{\text{after}}$ i.e. there is no significant difference in the blood pressure readings before and after the injection of the drug. The alternate hypothesis is $H_0 : \mu_{\text{before}} > \mu_{\text{after}}$ i.e. the stimulus resulted in an increase in the blood pressure of the patients. Taking $\alpha = 1\%$ and $\alpha = 5\%$, as **n = 13**, $\gamma = n - 1 = 12$, respective tabulated values are $t_{\text{tab}}\,|_{\alpha=1\% \text{ and } \gamma=12} = 3.055$ and $t_{\text{tab}}\,|_{\alpha=5\% \text{ and } \gamma=12} = 2.179$. Now, we compute the value of test criterion. For this, consider

x_i	$\overline{x}$	$x_i - \overline{x}$	$\left(x_i - \overline{x}\right)^2$
5		2	4

2		-1	1
8		5	25
-1		-4	16
3		0	0
0		-3	9
-2		-5	25
1	3	-2	4
5		2	4
0		-3	9
4		1	1
6		3	9
8		5	25
Total = 39		**-**	**Total = 132**

Consider $S = \sqrt{\dfrac{\sum_{i=1}^{i=n} \left(x_i - \overline{x}\right)^2}{n-1}} = \sqrt{\dfrac{132}{12}} = \sqrt{11} = 3.317$. Therefore, $t_{\text{cal}} = \left|\dfrac{\overline{x}-\mu}{S}\right| \cdot \sqrt{n}$ may be obtained as $t_{\text{cal}} = \left|\dfrac{0-3}{3.317}\right| \sqrt{13} = 3.2614$. As the calculated value 3.2614 is more than the tabulated values of 3.055 and 2.179, we accept the alternate hypothesis that after the drug is given to patients, there is an increase in the blood pressure level.

3. the life time of electric bulbs for a random sample of 10 from a large consignment gave the following data: 4.2, 4.6, 3.9, 4.1, 5.2, 3.8, 3.9, 4.3, 4.4, 5.6 (in '000 hours). Can we accept the hypothesis that the average life time of bulbs is 4, 000 hours?

Solution: Set up $H_0 : \mu = 4{,}000$ hours , $H_1 : \mu < 4{,}000$ hours . Let us choose that $\alpha = 5\%$. Then tabulated value is $t_{\text{tab}}\,|_{\alpha=5\% \text{ and } \gamma=9} = 2.262$. To find the test criterion, consider

x_i	$\overline{x}$	$x_i - \overline{x}$	$\left(x_i - \overline{x}\right)^2$

4.2		-0.2	0.04
4.6		0.2	0.04
3.9		-0.5	0.25
4.1		-0.3	0.09
5.2	**4.4**	0.8	0.64
3.8		-0.6	0.36
3.9		-0.5	0.25
4.3		-0.1	0.01
4.4		0.0	0.0
5.6		1.2	1.44
Total = 44		**-**	**Total = 3.12**

Consider $\boldsymbol{S} = \sqrt{\dfrac{\sum_{i=1}^{i=n} \left(x_i - \bar{x}\right)^2}{n-1}} = \sqrt{\dfrac{3.12}{9}} = 0.589$. Therefore, $t_{\text{cal}} = \left|\dfrac{\bar{x} - \mu}{S}\right| \cdot \sqrt{n}$ is computed as

$t_{\text{cal}} = \left|\dfrac{4.4 - 4.0}{0.589}\right| \cdot \sqrt{10}$ = 2.148. As the computed value is lower than the tabulated value of 2.262, we conclude that mean life of time bulbs is about 4, 000 hours.

A discussion on χ^2 test and Goodness of Fit

Recently, we have discussed t – distribution function (i.e. t – test). The study was based on the assumption that the samples were drawn from normally distributed populations, or, more accurately that the sample means were normally distributed. Since test required such an assumption about population parameters. For this reason, A test of this kind is called parametric test. There are situations in which it may not be possible to make any rigid assumption about the distribution of population from which one has to draw a sample.

Thus, there is a need to develop some non – parametric tests which does not require any assumptions about the population parameters.

With this in view, now we shall consider a discussion on χ^2 distribution which does not require any assumption with regard to the population. The test criterion corresponding to this distribution may be given as $\chi^2 = \dfrac{\sum_i O_i - E_i\ ^2}{E_i}$ where

- $E_i : \dfrac{RT \cdot CT}{N}$ O_i : Observed values**,** E_i : Expected values**.**

When Expected values are not given, one can calculate these by using the following relation; $E_i : \dfrac{RT \cdot CT}{N}$. Here, **RT** means the row total for the cell containing the row, **CT** is for the column total for the cell containing columns, and **N r**epresent the total number of observations in the problem.

The calculated χ^2 value (i.e. test criterion value or calculated value) is compared with the tabular value of χ^2 value for given degree of freedom at a certain prefixed level of significance. Whenever the calculated value is lower than the tabular value, we continue to accept the fact that there is not much significant difference between expected and observed results.
On the other hand, if the calculated value is found to be more than the value suggested in the table, then we have to conclude that there is a significant difference between observed and expected frequencies.

As usual, degrees of freedom are $\gamma = n - k$ where **k** denotes the number of independent constraints. Usually, it is 1 as we will be always testing null hypothesis against only one hypothesis, namely, alternate hypothesis.

This is an approximate test for relatively a large population.

For the usage of test, the following conditions must checked before employing the test. These are:

1. The sample observations should be independent.
2. Constraints on the cell frequencies, if any, must be linear.
3. i.e. the sum of all the observed values must match with the sum of all the expected values.
4. **N,** total frequency should be reasonably large
5. No theoretical frequency should be lower than 5.
6. It may be recalled this test is depends on χ^2 test: The set of observed and expected frequencies and on the degrees of freedom, it does not make any assumptions regarding the population.

ILLUSTRATIVE EXAMPLES

1. From the data given below about the treatment of 250 patients suffering from a disease, state whether new treatment is superior to the conventional test.

Data	Number of patients		
	Favourable	Not favorable	Total
New one	140	30	170
Conventional	60	20	80
Total	200	50	280

Solution: We set up null hypothesis as there is no significance in results due to the two procedures adopted. The alternate hypothesis may be assumed as there could be some difference in the results. Set up level of significance as $+\left(\frac{112-100}{100}{}^{2}\right)+\left(\frac{71-50}{50}{}^{2}\right)+\left(\frac{32-10}{10}{}^{2}\right)$ $\alpha = 5\%$ then tabulated value is $\chi^2 |_{\alpha=0.05,\gamma=1} = 3.841$.

Consider finding expected values given by the formula, $\text{Expectation(AB)} = \frac{\text{RT}\cdot\text{CT}}{\text{N}}$

where RT means that the row total for the row containing the cell, CT means that the total for the column containing the cell and N, total number of frequencies. Keeping these in view, we find that expected frequencies are

B

	136	34	170
A	64	16	80
	200	50	250

Note: $\frac{170\cdot 200}{250} = 136$; $\frac{170\cdot 50}{250} = 34$, $\frac{80\cdot 200}{250} = 64$ and $\frac{80\cdot 50}{250} = 16$.

O_i	E_i	$O_i - E_i$	$O_i - E_i{}^{2}$	$O_i - E_i{}^{2} / E_i$

140	136	4	16	0.118
60	64	-4	16	0.250
30	34	-4	16	0.471
20	16	4	16	1.000
			Total	1.839

As the calculated value 1.839 is lower than the tabulated value $\chi^2|_{\alpha=0.05,\gamma=1} = 3.841$, we accept the null hypothesis, namely, that there is not much significant difference between the two procedures.

2. A set of five similar coins is tossed 320 times and the result is

No. of heads	**0**	**1**	**2**	**3**	**4**	**5**
Frequency	**6**	**27**	**72**	**112**	**71**	**32**

Test the hypothesis that the data follow a binomial distribution function.

Solution: We shall set up the null hypothesis that data actually follows a binomial distribution. Then alternate hypothesis is, namely, data does not follow binomial distribution. Next, to set up a suitable level of significance, $\alpha = 5\%$, with **n** = 6, degrees of freedom is $\gamma = 5$. Therefore, the tabulated value is $\chi^2|_{\alpha=0.05,\gamma=5} = 11.07$. Before proceeding to finding test criterion, first we compute the various expected frequencies. As the data is set to be following binomial distribution, clearly probability density function is $b(n,p,k) = \binom{n}{k} p^k q^{n-k}$. Here, $n = 320$, $p = 0.5$, $q = 0.5$, and **k** takes the values right from 0 up to 5. Hence, the expected frequencies of getting 0, 1, 2, 3, 4, 5 heads are the successive terms of the binomial expansion of $320 \cdot (p+q)^5$. Thus, expected frequencies E_i are 10, 50, 100, 100, 50, 10. Consider the test criterion given by

$$\chi^2|_{\text{cal}} = \frac{\sum_i (O_i - E_i)^2}{E_i};$$

Here, observed values are: O_i : 6, 27, 72, 112, 71, 32

The expected values are: E_i : 10, 50, 100, 100, 50, 10. Consider

$$\chi^2|_{cal} = \left(\frac{6-10^{\ 2}}{10}\right) + \left(\frac{27-50^{\ 2}}{50}\right) + \left(\frac{72-100^{\ 2}}{100}\right) + \left(\frac{112-100^{\ 2}}{100}\right) + \left(\frac{71-50^{\ 2}}{50}\right) + \left(\frac{32-10^{\ 2}}{10}\right)$$

= 78.68. As the calculated value is very much higher than the tabulated value of 3.841, we reject the null hypothesis and accept the alternate hypothesis that data does not follow the binomial distribution.