X100/13/01

NATIONAL QUALIFICATIONS 2014

TUESDAY, 6 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1 Calculators may be used in this paper.

2 Candidates should answer **all** questions.

3 **Full credit will be given only where the solution contains appropriate working.**

Marks

Answer all the questions

1. (*a*) Given

$$f(x) = \frac{x^2 - 1}{x^2 + 1},$$

obtain $f'(x)$ and simplify your answer. **3**

(*b*) Differentiate $y = \tan^{-1}(3x^2)$. **3**

2. Write down and simplify the general term in the expression $\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}$.

Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. **5**

3. Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ.

$$\begin{aligned} x + y + z &= 2 \\ 4x + 3y - \lambda z &= 4 \\ 5x + 6y + 8z &= 11 \end{aligned}$$

For what values of λ does this system have a solution?

Determine the solution to this system of equations when $\lambda = 2$. **6**

4. Given $x = \ln(1 + t^2)$, $y = \ln(1 + 2t^2)$ use parametric differentiation to find $\frac{dy}{dx}$ in terms of t. **3**

5. Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ where

$$\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j},\ \ \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k},\ \ \boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}.$$

Calculate $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$. **3**

Interpret your result geometrically. **1**

Marks

6. Given $e^y = x^3\cos^2 x$, $x > 0$, show that

$$\frac{dy}{dx} = \frac{a}{x} + b\tan x, \text{ for some constants } a \text{ and } b.$$

State the values of a and b. **3**

7. Given A is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

prove by induction that

$$A^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \ n \geq 1.$$ **4**

8. Find the solution $y = f(x)$ to the differential equation

$$4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$$

given that $y = 4$ and $\frac{dy}{dx} = 3$ when $x = 0$. **6**

9. Give the first three non-zero terms of the Maclaurin series for $\cos 3x$. **2**

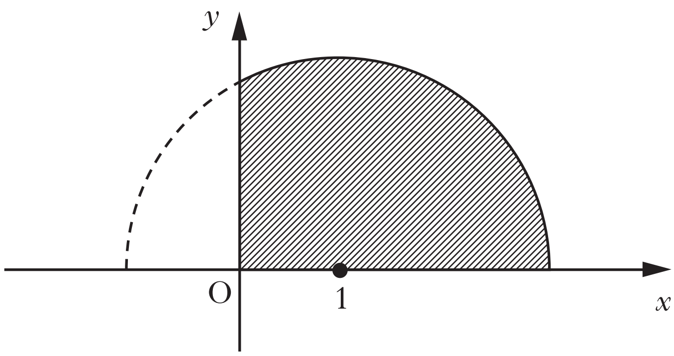
Write down the first four terms of the Maclaurin series for e^{2x}. **1**

Hence, or otherwise, determine the Maclaurin series for $e^{2x}\cos 3x$

up to, and including, the term in x^3. **3**

10. A semi-circle with centre (1, 0) and radius 2, lies on the x-axis as shown.

Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis. **5**

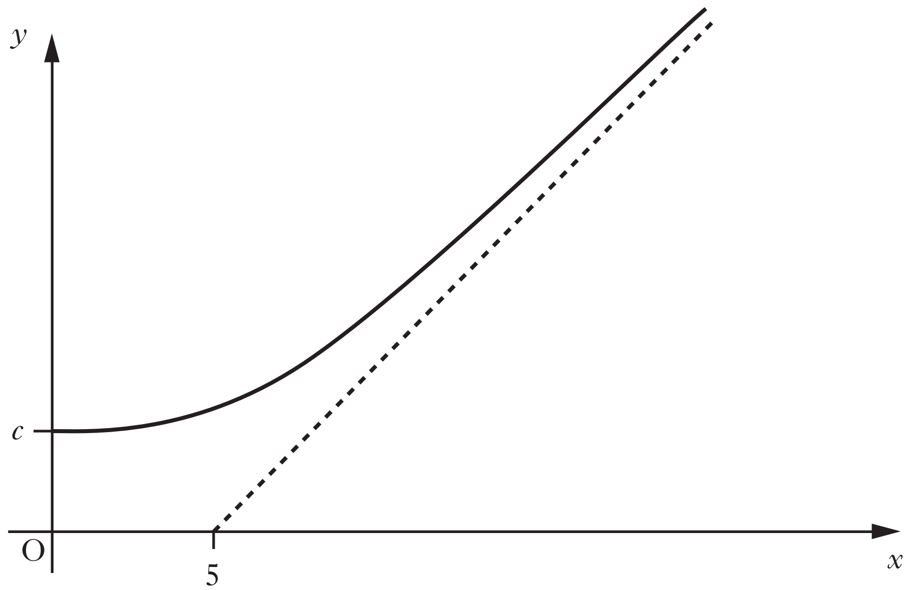


[Turn over

11. The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y-axis at $(0, c)$, where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Copy the above diagram.

On the same diagram, sketch the graph of $y = f^{-1}(x)$.

Clearly show any points of intersection and any asymptotes. **4**

(b) What is the equation of the asymptote of the graph of $y = f(x + 2)$? **1**

(c) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution? **1**

12. Use the substitution $x = \tan\theta$ to determine the exact value of

$$\int_0^1 \frac{dx}{\left(1+x^2\right)^{\frac{3}{2}}}.$$

6

13. The fuel efficiency, F, in km per litre, of a vehicle varies with its speed, s km per hour, and for a particular vehicle the relationship is thought to be

$$F = 15 + e^x(\sin x - \cos x - \sqrt{2}), \quad \text{where } x = \frac{\pi(s-40)}{80},$$

for speeds in the range $40 \leq s \leq 120$ km per hour.

What is the greatest and least efficiency over the range and at what speeds do they occur? **10**

Marks

14. (*a*) Given the series $1 + r + r^2 + r^3 + \ldots$, write down the sum to infinity when $|r| < 1$.

Hence obtain an infinite geometric series for $\dfrac{1}{2-3r}$.

For what values of r is this series valid? **4**

(*b*) Express $\dfrac{1}{3r^2 - 5r + 2}$ in partial fractions.

Hence, or otherwise, determine the first three terms of an infinite series

for $\dfrac{1}{3r^2 - 5r + 2}$.

For what values of r does the series converge? **6**

15. (*a*) Use integration by parts to obtain an expression for

$$\int e^x \cos x \, dx.$$ **4**

(*b*) Similarly, given $I_n = \int e^x \cos nx \, dx$ where $n \neq 0$,

obtain an expression for I_n. **4**

(*c*) Hence evaluate $\int_0^{\frac{\pi}{2}} e^x \cos 8x \, dx$. **2**

16. (*a*) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$. **3**

(*b*) Write down the four solutions to the equation $z^4 - 1 = 0$. **2**

(*c*) Plot the solutions of both equations on an Argand diagram. **1**

(*d*) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$. **2**

(*e*) Hence identify all the solutions to the equation

$$z^6 + z^4 + z^2 + 1 = 0.$$ **2**

[*END OF QUESTION PAPER*]

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